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“The measurement of the circle” of Archimedes in Naṣīr al-Dīn al-Ṭūsī’s revision of the ‘middle books’ (*taḥrīr al-mutawassiṭāt*)

'تكسير الدائرة' لارشميدس في كتاب تحرير المتوسطات لنصير الدين الطوسي

Bachelor thesis

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October, 2008

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Introduction

The influence of mathematicians of medieval Islamic civilization on their contemporary society and on the development of science in both the East and the West can only be understood after close study of the life and works of these mathematicians. How surprising is it then to see that even the most basic and widespread mathematical treatises are still either only available in manuscripts or severely understudied (or even worse, both!). An example of such an important work is the collection of the ‘middle books’ (*taḥrīr al-mutawassiṭāt*) of the savant Naṣīr al-Dīn al-Ṭūsī (generally acknowledged to be the most important Islamic scientist of the 13th century C.E.). This collection consists mostly of Arabic versions of Greek treatises (and to a lesser extent treatises of Arabic origin), reordered, rendered understandable again from defect translations, annotated with comments and put together with new insights. In all, it seems as a very important project and the large collection of extant manuscripts shows that it had a significant influence on mathematicians in the centuries to come. Despite all this, little is known on what these ‘middle books’ exactly are.

In this thesis I want to present my first footsteps into this exciting field , by studying *The measurement of the circle* of Archimedes as al-Ṭūsī has edited and commented it. *The measurement of the circle* is a treatise that belongs in the collection of the ‘middle books’. It was written by Archimedes (ca. 250 B.C.E.) and consists in its extant form of three propositions. The first one is that a circle is equal to a right triangle of which one of the sides is equal to the radius and the other equal to the perimeter. The second proposition is even more interesting, as it gives for the first time in history a very good approximation of π (Pi). The third relies on the second and gives $\frac{22}{7}$ as an approximation of π. Al-Ṭūsī not only edits the text, but includes also original contributions and completes or corrects Archimedes’ text here and there. This makes especially the comparison between Archimedes original text and al-Ṭūsī’s edition interesting.

From a student’s perspective it is also interesting to see that such an important topic is not very well documented by scholars. Maybe an explanation on the question why this has not been properly studied before is the sheer vastness of the subject. *The measurement of the circle* can be typed out in two pages, yet all kinds of interesting questions and issues arose to me while studying al-Ṭūsī’s rendering of the text. For example, what is exactly this collection of *al-mutawassiṭāt* people speak of? Why did al-Ṭūsī undertook a project to produce new editions of these texts? To what extend did he made original contributions? All these questions can be answered by an educated guess, but an answer firmly backed up by evidence is not as easily given as one would think. Again, we can make general assumptions but we just don't know for sure until we study the sources. This is also why besides trying to answer these questions, these thesis also provides the reader with a fully worked out text which students of Arabic should be able to read without any problem.

The thesis is divided into four chapters. In the first one a historical approach is taken; some background knowledge of Islamic civilization could come in handy but is not required. Al-Ṭūsī's life as a mathematician is described, the history of the 'middle books' is explained, the historical background of *The measurement of the circle* itself is given and it finishes by placing al-Ṭūsī's revision of *The measurement of the circle* in the context of his collection of the 'middle books'. The second chapter is a rendering of the full text of al-Ṭūsī. Next to the Arabic edition a translation is placed, and next to that is a translation of Archimedes' original text. The figures that are in the manuscript are also reproduced and placed after the text. Chapter three is devoted to a commentary on the text. It involves mathematical, historical, linguistical and other issues. Chapter four completes the thesis by presenting some final results and remarks on further research. A bibliography at the end contains full references to all publications that have been used and there is also a table of contents of the Tehran facsimile edition appended.

In the hope to inspire other people to pursue research in this fascinating field,

Eric van Lit

October, 2008

Chapter 1

Naṣīr al-Dīn al-Ṭūsī's life and his mathematical contributions[1]

The scientist who edited the text under discussion here is Abū Jaᶜfar Muḥammad ibn Muḥammad ibn al-Ḥasan Naṣīr al-Dīn al-Ṭūsī, for obvious reasons usually referred to as Naṣīr al-Dīn al-Ṭūsī (I will even abbreviate that to simply 'al-Ṭūsī' from now on). Al-Ṭūsī lived from 597/1201[2] (Ṭūs, Khurāsān) to 672/1274 (Baghdād) and gained widespread popularity already during his own life, eventually giving him a whole range of honorary titles of which *khʷāja* (distinguished scholar), *ustādh al-bashar* (teacher of mankind) and *al-muᶜallim al-thālith* (the third teacher)[3] are among the most honorable. Al-Ṭūsī was a prolific writer, completing at least 150 different works on topics ranging from theology and religious law to ethics and astronomy and everything in between. As al-Ṭūsī's life was one of turmoil (because of the Mongol invasion) and political and religious ambiguities (switching back and forth to various political and religious stances) it is not easy to give a straightforward biography. However, concerning his life as a mathematician, I believe a somewhat unequivocal account can be given.

Al-Ṭūsī was born into a family that seemed to be rather sympathetic towards the rational sciences (like mathematics, logic and medicine). At first his father taught him the traditional Islamic sciences (*Qur'ān*, *ḥadīth* and *fiqh*), but presumable already at very young age (10-12)[4] al-Ṭūsī also began to study topics like mathematics and (Greek) philosophy. His mathematical education can be divided into three stages. Towards his puberty he studied under supervision of Kamāl al-Dīn Muḥammad al-Ḥāsib, focusing explicitly on mathematics. [5] Although we are not sure, he will probably covered the *Elements* of Euclid in this period. The second stage is between 610/1213 and 618/1221, when he left for Nīsābūr to study under supervision of Quṭb al-Dīn al-Miṣrī and Farīd al-Dīn Dāmādh. Here he studied a variety of subjects and we cannot be exactly sure to what extent he received mathematical

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[1] This biography draws from Daiber, H., *al-Ṭūsī, Naṣīr al-Dīn*, EI², 746a-752a and Ragep, F.J., *Naṣīr al-Dīn al-Ṭūsī's* memoir on astronomy, Springer-Verlag, New York, 1993, pp. 3-23

[2] The first number of all dates indicate the Hijrī date, according to the Islamic calendar (which started in 622 C.E. and is lunar). The second number refers to C.E.: Common Era.

[3] The 'first teacher' being Aristotle and the 'second teacher' al-Fārābī.

[4] Ragep, pp. 6, asserts that al-Ṭūsī left for Nīsābūr "sometime after 610/1213", this makes an age of 12. As it is not uncommon that someone can learn a great deal of the Islamic sciences on a reasonable level already at an age of 10 (this should especially be not difficult for a bright mind as al-Ṭūsī), it would leave a gap of a couple of years between his Islamic studies and his trip to Nīsābūr. For a similar quick learner, see: Gutas, D., *Avicenna and the Aristotelian Tradition*, E.J. Brill, Leiden, 1988, p. 152. It should be noted that al-Ṭūsī studied and wrote on religious and theological topics throughout his life, see also his spiritual autobiography: Badakhchani, S.J., *Contemplation and Action*, I.B. Tauris, London, 1998

[5] Badakhchani, S.J., p. 26

training. It seems probable that he at least covered some mathematical texts of for example Archimedes or Menelaus (in all, what is referred to as the 'middle books', see below *The book "taḥrīr al-mutawassiṭāt"*) which were all available in Arabic translations. Whether he had to flee for the Mongol invasion army, or it was just his thriving for more knowledge, al-Ṭūsī left for Iraq and became a student of Kamāl al-Dīn ibn Yūnus. Ibn Yūnus was a famous mathematician and astronomer who had a good understanding of Ptolemy's *Almagest*, and there are two reasons to believe that al-Ṭūsī studied the *Almagest* with him. First of all, students back then wanted to complete their mathematical training with the *Almagest* in order to become an astronomer (which held better career opportunities). Secondly, ibn Yūnus was a Sunnī muslim, trained in the Shāfiᶜite school of religious law, so it seems unlikely that al-Ṭūsī (born and raised a Shīᶜī) would have studied religious topics with him. After this period he traveled in 630/1233 to Qūhistān to begin his professional career.

This makes the following timeline:

- Age 10-13: Euclid's *Elements* under supervision of Kamāl al-Dīn Muḥammad al-Ḥāsib.
- Age 13-20: Various mathematical treatises ("middle books") under supervision of Quṭb al-Dīn al-Miṣrī and Farīd al-Dīn Dāmādh.
- Age 20-30: Ptolemy's *Almagest* under supervision of Kamāl al-Dīn ibn Yūnus.

His professional career first took off under Ismāᶜīlī patronage. Later, when the Mongols definitively took over power, he sought refuge with them, giving him a second career at the Mongol court. His ease of change of political stance and his dubious role in the fall of the Ismāᶜīlī empire makes him suspect on his actual stance towards the Ismāᶜīlīs in the first place and this is further promoted by the rumor that he secretly corresponded with the caliph in Baghdād while staying with the Ismāᶜīlīs. This in turn promotes the story that the Ismāᶜīlīs found out and held him in captivity from somewhere around 644/1246 to 653/1255 in the fortress stronghold of Alamūt.[6] It does seem romantic to picture the genius al-Ṭūsī held against his own will in a big fortress, writing Ismāᶜīlī texts he himself did not agree with. However, we just don't know for sure if he himself was an Ismāᶜīlī or not let alone if he was held in captivity.

We do know that Alamūt had one of the biggest libraries in a very large region and so it is to no surprise that here al-Ṭūsī's writing career really got under way. This is also the place al-Ṭūsī begins to write revisions of Greek mathematical treatises in Arabic translation and works by Islamic mathematicians (what eventually would culminate in his *taḥrīr al-mutawassiṭāt*). A list of his publications is given below.

[6] For more information see Daiber, pp. 746b and Ragep, pp. 9-13

After the sack of Alamūt al-Ṭūsī travelled with the Mongol army to Baghdād, becoming a trusted advisor and administrator of *waqf* (religious endowments). With *waqf* money, he was able to build an observatory in Marāgha. This was a big project in which all kinds of astronomical instruments wre built and a vast library was established, even attracting astronomers from China. Al-Ṭūsī became the director of the observatory and produced astronomical tables in a work called *Zīj-i Īlkhānī*. Besides that he also produced the famous *al-Tadhkira fī ᶜilm al-hay' a* (Memoir on Astronomy) in which he tried to improve on the Ptolemaic model of the cosmos.[7] For unknown reasons he left for Baghdād in 672/1274, in this same year he died and was buried presumably near the shrine of Mūsā al-Kāẓim (*al-Kāẓimayn*).

Al-Ṭūsī's work seemed to be very popular in the centuries after his death and even today there is a very large number of manuscripts preserved in libraries all around the world. Unfortunately, there is not a lot of research on his mathematical writings but combining Max Krause's *Stambuler Handschriften islamischer Mathematiker* and the catalogue of Rosenfeld and Ihsanoglu the following list of treatises could be established.[8]

The following are all revisions or commentaries on Greek works (in Arabic translation) or Arabic works:

- "Revision of the Book *Almagest*" of Ptolemaeus — shawwāl 644/February 1247
- "Revision of the Book *Elements*" of Euclid — 22 sha'bān 646/10 December 1248
- "Revision of the Book *Optica*" of Euclid — 13 shawwāl 651/6 December 1253
- "Revision of the Book *On the Moving Sphere*" of Autolycus — 651/1253-54
- "Revision of the Book *Spherics*" of Theodosius — jumādā 651/August 1253
- "Revision of the Book *Phenomena*" of Euclid — 10 rabī' II 653/19 May 1255
- "Revision of the Book *Data*" of Thābit ibn Qurra — 653/1255
- "Revision of the *Book of knowledge on Measuring Plane and Spherical Figures*" by the Banū Mūsā — 653/1255
- "Revision of the Book *Lemmas*" of Archimedes — 653/1255
- "Revision of the Book *On Days and Nights*" of Theodosius — 653/1255

[7] An edition and translation of this work can be found in Ragep's book.

[8] Krause, M., "Stambuler Handschriften islamischer Mathematiker", *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* (Berlin), Abteilung B: Studien 3. 1936, pp. 437-532. Reprinted in: Sezgin, F., *Miscellaneous Texts and Studies on Islamic Mathematics and Astronomy VIII*, vol. 83 of Islamic Mathematics and Astronomy, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp.237-332. Krause gives for some manuscripts dates based on textual evidence.
Rosenfeld, B.A., Ihsanoglu, E., *Mathematicians, Astronomers, and other scholars of Islamic civilization and their works (7th-19th c.)*, Research Centre for Islamic History Art and Culture, Istanbul, 2003, pp. 211-219

- “Revision of the Book *On the Ascension of Stars*” of Hypsicles 653/1255
- “Revision of the Book *On Risings and Settings*” of Autolycus 653/1255
- “Revision of the Book *On the Sizes and Distances of the Sun and Moon*” of Aristarch 658/1259-60
- “Revision of the Book *On the Sphere and the Cylinder* “ and “Revision of the Book *Measurement of the Circle*” of Archimedes 661/1262-63
- “Revision of the Book *Spherics*” of Menelaus Shab’ān 663/May 1265[9]

- “Revision of the Book *Data*” of Euclid
- “Revision of the *Book of Assumptions*” of Thābit ibn Qurra
- “Revision of the Book *Conic Sections*” of Apollonius
- “Revision of the Book *On Habitations*” of Theodosius
- On Premises of the Work “Conic Sections” of Apollonius

The following are his own, original contributions:

- Removal of the Veil from Mysteries of Secants
- Treatise on Secants in the Science of Geometry
- Treatise on Salvation from Doubts about Parallel Lines
- Collection of Arithmetic by Means of Board and Dust
- Book on Multiplication and Division
- Treatise on Arithmetic Problems and Algebra and Almucabala
- Treatise on Proving the Impossibility of a Square Number being the Sum of two Odd Square Numbers to be a Square Number
- Commentary on “Propositions of Substantiation”
- Comments to Euclid
- Book of Victory in Algebra and Almucabala
- On Motion of Rolling and Ratio between Straight and Curved Lines
- Projecting the Sphere onto a Plane
- Inheritance According to the Opinion of Ahl al-Bayt

As Rosenfeld/Ihsanoglu’s references are not always entirely clear, there could be in some way overlap in dated and undated treatises and it could also very well be that this is not the complete list of al-Ṭūsī’s mathematical contributions. The big picture however, is quite clear. Al-Ṭūsī spent at

[9] Krause also mentions “*Centiloquiums* of Ptolemaus 20 rajab 663/ 8 May 1265” but this does not seem to be a mathematical treatise (rather, an astrological one).

least from 644/1247 to 663/1265 time on mathematics, especially preparing new editions (with commentary) of Arabic translations of Greek texts. It is noteworthy to see that he began in reverse order; the first dated treatise is the *Almagest*. Only a year later he published the other end of the spectrum (i.e. the book for beginners rather than more advanced students) in the form of the *Elements*. A suggestion for this choice could be that he found them the most important. Then there seems to be a five year gap, in which he undoubtedly focused more on (Ismaᶜīlī) philosophy and theology. From 651/1253 onward he frequently finished a mathematical text, I will come back on it in the next paragraph. Note that the year 653 also covers some of the year 1256 C.E., but considering the precarious situation with the Mongol army knocking on the door of the Alamūt fortress, it seems more likely that he tried to finish as many treatises as possible before the Mongols actually took control over Alamūt and lived up to their reputation of destroying everything they could.[10] Also his active role in the negotiations between the Ismaᶜīlīs and the Mongol army that year probably kept him away from his scholarly activity. After the sack of Alamūt we again notice a gap of some five years. In these years he worked for Hülegü, the Mongol emperor. Only when he is at safe distance, when he directs the constructing of the Marāgha observatory, did he pick up the commentaries of the Greek texts again. After 663/1265 we lose track of his mathematical activity. Maybe he felt he finished his revision project, maybe he spend more time on astronomy at the observatory and maybe (but less likely) he continued revising Greek texts either way.

Ragep states at the beginning of his biography of al-Ṭūsī that his "Hellenism led him to pursue knowledge for its own sake"[11] and even adds a footnote stating that "this *should* be obvious; however the insistence of numerous modern commentators that Islamic scientists could never rise to the Greek view that knowledge was to be pursued for its own sake compels one to state the obvious."[12] This overview of al-Ṭūsī as a mathematician indeed shows his deep, unselfish commitment to mathematics. Already at a young age he gave mathematics a try and soon after, only a child of 12 or 13 years old, he left for Nīsābūr to seek new teachers. Even better, after that he traveled as far as Mosul and Baghdād to become a student of Kamāl al-Dīn ibn Yūnus. To this point, it could still be that al-Ṭūsī was just in for the astronomy and indeed, he made quite some fame as an astronomer. As an astronomer who also wrote extensively on subjects such as philosophy, ethics and (Ismāᶜīlī) theology, it seems as if he had more than enough on his mind. Yet still he managed to make time to write commentaries on important mathematical treatises. Especially his life-long project on the 'middle books', that even continued while he prepared and directed the prestigious observatory at Marāgha, proofs that he felt deeply committed to mathematics for no other reason than mathematics itself.

[10] It would have been a smart anticipation of al-Ṭūsī as Juwaynī mentions that except for a few manuscripts the whole library of Alamūt was destroyed. Ragep, p. 19
[11] Ragep, pp. 4
[12] Ragep, pp. 4 note 7

The *taḥrīr al-mutawassiṭāt*

The title 'taḥrīr al-mutawassiṭāt' consists of two words who deserve separate attention. First there is *taḥrīr*, which usually means 'liberation', but can be used in this context as the word for 'commentary', 'elaboration' or 'exposition'. Commentaries are quite common in Arabic medieval times, usually scientists worked at the court of a kingdom or caliphate and wrote treatises on questions the scientist was asked. For example, the famous Nasirean Ethics (*Akhlāq-i Nāṣirī*) is not called Nasirean because of al-Ṭūsī himself, but in honour of Muḥtashim Nāṣir al-Dīn, an Isma^cīlī governor he worked for. The mathematical editions under discussion here miss these dedications, but start by mentioning the title of the text and the name of the author which is about to be rendered and commented on. If Al-Ṭūsī indeed did not wrote these treatises on request he must have found it really important to complete all these editions, as he might as well could have written on other subjects for which he did get paid, and not waste precious paper, ink and (most of all) time on this 'voluntary work'.

The term *mutawassiṭāt* refers to a collection of Arabic translations of Greek mathematical treatises. Extremely little is known about this term, the best study of it going back to the year 1865.[13] From this study we know that the term was used as early as the 10th century C.E.[14] In the centuries after that, the term was used as if the collection were one book in contrast to the *Elements* and the *Almagest*.[15] It is rather unlikely that the term used to refer for example to their mediocre popularity, rather it seems that the most probable use is of the books that come didactically between the *Elements* and the *Almagest*.[16] For example, al-Nasawī states at the beginning of his rendering of (pseudo-)Archimedes' *Lemmas*; "... the middle books [...] which it is necessary to read between the book of Euclid and the *Almagest*."[17] As it became used as an didactical term over the years, so it also included treatises by Islamic authors rather than only Arabic translations of Greek treatises. Combining the list of Steinschneider[18] and the table of contents of the facsimile of al-Ṭūsī (see appendix), the following list of "middle books" can be made:

- Euclid, *Data*,
- Euclid, *Optics*
- Euclid, *Phaenomena*
- Autolycus, *Moving Sphere*

[13] Steinschneider, M., "Die mittleren Bücher der Araber und ihre Bearbeiter", *Zeitschrift für Mathematik und Physik*, nr. 10, Leipzip, 1865, pp. 456-498. Reprinted in: Sezgin, F., *Euclid in the Arabic Tradition Text and Studies I*, Islamic Mathematics and Astronomy, vol. 17, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1997, pp. 54-97.
[14] Ibid., p. 56
[15] Ibid., pp. 56/57, note 11
[16] Ibid., pp. 57/58
[17] Ibid., p. 78
[18] Ibid., p. 65

- Autolycus, *Risings and Settings*
- Archimedes, *On the Sphere and Cylinder*, with the commentary of Eutocius
- Archimedes, *Measurement of the Circle*
- Theodosius, *Spherics*
- Theodosius, *Inhabited Places*
- Theodosius, *Days and Nights*
- Aristarchus, *Sizes and Distances of the Sun and Moon*
- Hypsicles, *Ascensions*
- Pseudo-Archimedes, *Lemmata*
- Menelaus, *Spherics*
- Abū Sahl Kūhī, *Additions to the Book On the Sphere and Cylinder of Archimedes*
- Thābit ibn Qurra, *Assumed Things*
- Thābit ibn Qurra, *Data*
- Thābit ibn Qurra, *Sector of the Figure*
- Muḥammad ibn Mūsā (Banū Mūsā), *Measurement of the figure*
- Ibn al-Haytham, *Division of the Line which Archimedes used in the second Book On the Sphere and Cylinder*

This list serves as an indication of what kind of treatises can be expected in the 'middle books'. In the manuscripts one can encounter various combinations of these texts, it is for example also reported that one manuscripts of al-Ṭūsī's 'middle books' included al-Ṭūsī's famous essay on Euclid's parallel postulate and the *Correction on Optics* from al-Kindī.

Taking the words *taḥrīr* and *mutawassiṭāt* together gives us the project al-Ṭūsī started. We don't know if there were similar projects done before. Al-Ṭūsī states at the beginning of his commentary of the *Sphere and Cylinder* of Archimedes that he read it first from a poor copy of Thābit ibn Qurra and later in an old codex of Ishāq ibn Ḥunayn. He then states:

"In this codex I found what I sought, and so I came to the idea of preparing the book in order, setting out its contents precisely, proving its postulates, adding its necessary premises, and transmitting a commentary on whatever of it is unclear, based on what I could take from Eutocius and other representatives of this discipline [...] This aim have I fulfilled. Then I have added at the end of the book the writing of Archimedes on the quadrature of the circle, since this depends on axioms which are brought up in the former work."[19]

[19] Sezgin, F., *Geschichte des Arabischen Schrifttums*, Band V Mathematik bis ca. 430H, Brill, Leiden, 1974, pp. 128-129. Translated in: Knorr. W.R., *Textual Studies in Ancient and Medieval Geometry*, Birkhäuser, Boston, 1989, p. 547

This establishes that al-Ṭūsī undertook his revision for at least the *Sphere and the Cylinder* primarily to prepare a solid text and render a clear understanding of the treatises he felt were important. It could be argued that al-Ṭūsī held equal reasons to revise other mathematical treatises. Unfortunately, we don't know exactly how influential al-Ṭūsī's *taḥrīr al-mutawassiṭāt* was in the Islamic tradition. The 11th to 17th century C.E. are pretty much unexplored territory so we can only assume, based on the excellence of al-Ṭūsī's text and on the very large number of manuscripts of the *taḥrīr al-mutawassiṭāt* that have survived, that al-Ṭūsī's text virtually replaced all older translations and became some sort of standard work indeed to be read (as a single book rather than as a collection of treatises) between the *Elements* and the *Almagest*.

Archimedes and the translation of *The measurement of the circle*

The measurement of the circle was originally written by Archimedes, who lived in the 3rd century B.C.E. The structure, its linguistics and the content tell us that it did not come down to us in the original form. For first of all the treatise is very short, consisting of only three propositions. Secondly, the second proposition cannot have been placed in the original version in the second place as it relies on the result of the third proposition. Besides these structural arguments, the text reveals very little of the Doric dialect Archimedes was used to write in.[20] The content itself is very short. On several occasions Archimedes merely states the result and leaves it to the reader to check the reasoning. One of the most striking issues is the ease at which Archimedes seemed to use the approximation of the root of 3, namely: $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$. According to Heath, the best solution at how Archimedes came to use these inequalities is the formula $a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}$.[21] However, for this formula to work you would have to use $a = \frac{34450}{19891}$ $(\approx\ 1.73194), b = \frac{461365}{1186955643} (\approx\ 0.000388696)$ which looks rather odd and raises the question how you would find those fractions. Unless we find historical evidence, we can only guess how Archimedes exactly calculated the inequalities.

We do not know much on the reception into the Arabic sciences. We at least know that *The measurement of the circle* was available before 857 C.E., as al-Kindī supposed to have corresponded on its content with the scientist Ibn Māsawayh (who died in 242/857).[22] On the original translation is still uncertainty, although the text was most probably translated first by Isḥāq ibn Ḥunayn and afterwards retranslated or improved by Thābit ibn Qurrah.[23] In his study of al-Kindī's treatise,

[20] Heath, T., *A History of Greek Mathematics*, vol. II, Clarendon Press, Oxford, 1965, p. 50 compare: Dijksterhuis, E.J., Archimedes, transl. by Dikshoorn, C., Princeton University Press, New Jersey, 1987, p. 222
[21] Heath T., p. 51
[22] Rashed, R., *Al-Kindī's Commentary on Archimedes' 'The Measurement of the Circle'*, pp. 7-54 in: *Arabic sciences and Philosophy*, vol. 3, Cambridge University Press, Cambridge, 1993, p. 15
[23] Sezgin, F., *GAS*, pp. 128-131

Rashed suspects there might have been another translation by Quṣṭā ibn Lūqā.[24] He also asserts in two footnotes that while the history of these translations has not yet been written, he examined all of the extinct manuscripts and will publish the results of the research on it.[25] Unfortunately, I did not come across a publication of him on this topic. The best research is the 500-page chapter of Knorr who gives an even wider overview of *The measurement of the circle* through the ages.[26]

The measurement of the circle is a text at which you can see al-Ṭūsī at work in full, giving proper commentaries on the statements Archimedes makes, and elaborating an expanding on it. In this thesis, al-Ṭūsī's revision on *The measurement of the circle* is studied mostly in comparison with Archimedes' original text. As it is of course also very interesting to track the development of the text over the centuries in Latin Europe, you could take a look at Clagett's very extensive documentation of the medieval Latin translations of the text. At the translation of Gerard of Cremona, a critical apparatus also indicates al-Ṭūsī's additions.[27]

Taksīr al-dā'irah within the *taḥrīr al-mutawassiṭāt*

It seems that al-Ṭūsī wrote his commentary on *The measurement of the circle* rather late (661/1262-63). However, he did write the commentary on Menelaus' *Spherics* later (663/1265), which is included before *The measurement of the circle* in the *taḥrīr al-mutawassiṭāt*. This means the actual compilation of the *taḥrīr al-mutawassiṭāt* was either conducted by al-Ṭūsī at the very end of his life or, more probable, after he had died. In any case, *The measurement of the circle* is the last book in the manuscripts I looked at and appears right after *On the Sphere and Cylinder*, also by Archimedes. In fact, as we saw already in the citation from his preface to the *On the Sphere and Cylinder*, *The measurement of the circle* is rather an appendix to *On the Sphere and Cylinder*. On the other hand, the lay-out of *The measurement of the circle* looks like it is a text on its own. The commentary of *On the Sphere and Cylinder* is properly concluded, including a proper Islamic formula and the new text is properly introduced (and likewise properly concluded). On the other hand, the text does not start with the *basmallah*,[28] which is a strong indication it was not meant to be read separately from *On the Sphere and Cylinder* (which does start with the *basmallah*).

[24] Rashed, R., pp. 15-16
[25] Rashed, R., p. 12 note 15 and p. 18 note 30
[26] Knor. W.R., Part III, The Textual Tradition of Archimedes' *Dimension of the Circle*
[27] Clagett, M., *Archimedes in the Middle Ages*, Vol. I The Arabo-Latin Tradition, The University of Wisconsin Press, Madison, 1964, pp. 40-55
[28] The *basmallah* is the phrase "in the name of God, the most compassionate, the most merciful". Virtually all texts from Islamic civilization start with this formula.

Chapter 2

Formal remarks

Al-Ṭūsī's revision of *The measurement of the circle* consists of the same three propositions Archimedes wrote, but in another order. He realized that the second proposition depended on the third and so swapped their order. At the end of proposition one, he proposes a generalization and another significant contribution is the elaboration of another approximation of π, making use of astronomical tables. The treatise is written on 5 folia and contains 5 illustrations.

The following table shows the transliteration of the labels in the geometrical figures that are used in the treatise.

Usage of labels									
Arabic	ا	ب	ج	د	ه	ز	ح	ط	ي
Greek	Α	Β	Γ	Δ	Ε	Ζ	Η	Θ	Ι
Transliteration	a	b	g	d	e	z	h	T	y

Usage of labels								
Arabic	ك	ل	م	ن	ص	ع	ق	س
Greek	Κ	Λ	Μ	Ν	Ϙ	Ο	Ρ	Ξ
Transliteration	k	l	m	n	x	o	q	s

In the second part of the second proposition, al-Ṭūsī makes use of sexagesimal numbers. These numbers are formed with letters which can for example be looked up in Toomer's edition of the Arabic translation of the Conics of Appolonius.[29] In the translated text they take the form of for example: 376; 59, 10, 59 which should be read as $376 + \frac{59}{60} + \frac{10}{60^2} + \frac{59}{60^3}$ (≈ 376,986).

The sources consulted

For al-Ṭūsī's revision of *The measurement of the circle* you fortunately do not have to restrict yourself to the usage of manuscripts. There is a printed edition of the text in *majmūᶜ al-rasā'il* (2 vols.), Ḥaydarābād, 1359/1940 , which is reprinted in Volume 48 of the *Islamic Mathematics and Astronomy* series.[30] In this edition it functions as an appendix of the larger "book on the sphere and cylinder

[29] Toomer, G.J., *Conics. Books V to VII : the Arabic translation of the lost Greek original in the version of the Banū Mūsā*, Springer Verlag, New York, 1990, p. xcii

[30] aṭ-Ṭūsī, Naṣīraddīn, 'Kitāb al-Kura wa-l-usṭuwāna li-Arkhimīdis [Archimedes] bi-taḥrīr', *majmūᶜ al-rasā'il* (2 vols.), Ḥaydarābād, 1359/1940, pp. 127-133. Reprinted in: Sezgin, F., *A Collection of Mathematical and Astronomical Treatises as Revised by Naṣīraddīn aṭ-Ṭūsī*, Volume 48 of *Islamic Mathematics and Astronomy*, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp. 377-389 (٣٧٧ – ٣٨٩)

with commentary" (*kitāb al-kura wa-l-usṭuwāna li-Arkhimīdis bi-taḥrīr*), other texts in this book include revisions on Autolycos, Aristarchos, Hypsicles and another text of Archimedes. For a better understanding of the text and to correct the Hyderabad edition, I studied a facsimile printed in Tehran.[31] This is a facsimile of Ms. Tabriz, Melli Library, no. 3484. Knorr published a translation that is based on the Hyderabad publication and ms. arab. 2467, Bibliothèque Nationale, Paris. While his main purpose is a linguistically one, his translation has some quirks and errors so I did not pay specific attention to his edition for my Arabic version and my translation (see Chapter 3, some of his mathematical observations regarding sexagesimal numbers are useful).[32] I used some preliminary notes on technical vocabulary by Lorch. These notes proved helpful to get started.[33]

I have tried to make an edited Arabic text out of both Arabic sources. Whenever one of the texts deviates a footnote shows the deviation with H for the Hyderabad-edition and T for the Tehran-edition. The translation of Archimedes' Greek text was taken from Thomas[34] with minor changes inspired by Dijksterhuis.[35]

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[31] al-Tusi, Nasir al-Din, *Taḥrīr-e Mutawassīṭāt*, Introduction by Dr. Jafar Aghayanī-Chavoshī, Insitute for Humanities and Cultural Studies, Tehran, 2005. Note the spelling error in the title (*mutawassīṭāt* rather than *mutawassiṭāt*).
[32] Knor. W.R., pp. 535-594
[33] Muwafi, A., Philoppou, A.N., *Appendix to 'An Arabic Version of Eratosthenes on Mean Proportionals'*, Journal of the History of Arabic Science 5. Aleppo, 1998, pp. 145-165. Reprinted in: Lorch, R., *Arabic Mathematical Sciences*, Variorum, Aldershot, 1995, part IV: *A Note on the Technical Vocabulary in Erastosthenes' Tract on Mean Proportionals*
[34] Thomas, I., *Selections Illustrating the History of Greek Mathematics*, vol. I From Thales to Euclid, Cambridge University Press, Cambridge, Massachusetts, 1967
[35] Dijksterhuis, E.J.

Bilingual text of *Taksīr al-dā'irah* together with Archimedes' text

Translation of Archimedes'text	*Translation of al-Ṭūsī's text*	*Arabic text of Naṣīr al-Dīn al-Ṭūsī*
Archimedes, Measurement of a circle	Essay of Archimedes on the measurement of the circle, it is three propositions.	مقالة ارشميدس في تكسير الدائرة وهي ثلاثة اشكال
Any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the base is equal to the circumference.	Every circle is equivalent to a right triangle of which one of the two sides which surround the right angle is equal to the half of the diameter of that circle, and the other [side] is equal to its perimeter. The result is that it is equal to the rectangle contained by half of its diameter and the line that is equal to half of its perimeter.	كل دائرة فهي مساوية لمثلث قائم الزاوية يكون احد ضلعيه المحيطين بالزاوية القائمة مساويا لنصف قطر تلك الدائرة والثاني مساوياً[36] لمحيطها والحاصل انها يساوي[37] سطح نصف قطرها في الخط المساوي لنصف محيطها
Let the circle **ΑΒΓΔ** have to the triangle **Ε** the stated relation; I say that is equal.	So let the circle be a circle **a b g d** and the said triangle a triangle **e**. If the circle is not equal to it, then it is either bigger or smaller.	فلتكن الدائرة دائرة[38] ا ب ج د والمثلث المذكور مثلث ه. فان لم تكن الدائرة مساوية له فهي إما اعظم منه وإما اصغر
For if possible, let the circle be greater, and let the square **ΑΓ** be inscribed, and let the arcs be	Let it be bigger first. We draw in the circle a square **a g**. It separates from it a bigger part than	وليكن اولا اعظم ونرسم في الدائرة مربع ا ج[39] وهو يفصل منها اعظم من نصفها وننصف ا ب

[36] H: مساويا
[37] H: تساوى
[38] H: دايرة

divided into equal parts, and let the segments be less than the excess by which the circle exceeds the triangle.

the half of it [i.e. the circle]. We bisect **a b** at **f** and similarly the four arcs and we connect the cords. So the resulting triangles separate more than half of the sections as has been proved before. This is repeated until there remains sections of the circle that are smaller than the measure of the excess of the circle over the triangle **e**.

على ف وهكذا القسي الاربع و نصل الاوتار فنفصل المثلثات الحادثة اعظم من نصف القطع لما مر بيانه وهاكذا مرة بعد اخرى الى ان تبقى من الدائرة قطع هي اصغر من مقدار زيادة الدائرة على مثلث ه

The rectilineal figure is therefore greater than the triangle.

So the equilateral figure that is in the circle is [then] bigger than the triangle.

فيكون الشكل المتساوي الاضلاع الذي في الدائرة اعظم من المثلث

Let **N** be the centre, and **NΞ** perpendicular;

Let the centre be **n** and we draw from there to one of the sides a perpendicular line, let it be **n s**.

وليكن المركز ن ونخرج منه على احد الاضلاع عمودا[40] وليكن ن س

NΞ is then less than the side of the triangle.

Then it is less than the line **n x** which is equal to one of the sides of the triangle **e** [i.e. the radius].

وهو اصغر من ن ص المساوي لاحد ضلعي مثلث ه

But the perimeter of the rectilineal figure is therefore less than the triangle **E**;

The perimeter of the equilateral figure is less than the perimeter of the circle [which] is equal to the other side of the triangle **e**. So the rectangle contained by **n s** times the perimeter of the figure, I mean, twice the magnitude of the figure is less than twice the triangle. So the figure

ومحيط الشكل المتساوي الاضلاع اصغر من محيط الدائرة المساوي[41] للضلع الآخر من مثلث ه فسطح ن س في محيط الشكل اعني ضعف مقدار الشكل اصغر من ضعف المثلث فالشكل اصغر من مثلث وكان اعظم منه

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[39] H: ا ب ج

[40] T: the sentence "وليكن المركز ... الاضلاع عمودا" is missing.

[41] H: المتساوي

which is absurd.	is less than the triangle and it is greater than it. This is a contradiction.	هذا خلف
Let the circle be, if possible, less than the triangle **E**, and let the square be circumscribed,	Then let the circle be smaller than the triangle and we draw around it a square **o q**. [The circle] separates from the square more than half of it.	ثم لتكن الدائرة اصغر من المثلث ونرسم عليها مربع ع ق فهي تفصل من المربع اعظم من نصفه
and let the arcs be divided into equal parts, and through the points let the tangents be drawn; the angle **OAP** is therefore right.	We bisect the arc **b a** at **f** and draw **z f T**, tangent to the circle at **f**. The radius **n f** is perpendicular to it [i.e. the tangent line]. We do this likewise with the other arcs.	وننصف[42] قوس ب ا على ف ونخرج ز ف ط مماساً[43] للدائرة على ف ويكون نصف قطر ن ف عمودا عليه وهكذا نعمل في سائر القسي
Therefore **OP** is greater than **MP**; for **PM** is equal to **PA**;	Because **q b** and **q a** are equal to each other and likewise **T b**, **T f**, **z f** and **z a**, [all] four are equal to each other. But **T q** and **q z** are equal. The two are together longer than **T z**. So **q T** is longer than **b T**,	ولأن ق ب ق ا متساويان وكذلك ط ب ط ف ز ف ز ا الاربعة متساوية يكون ط ق ق ز متساويين وهما معاً[44] اطول من ط ز فق ط[45] اطول من ب ط
and the triangle **PΟΠ** is greater than half the figure **OZAM**.	so triangle **q f T** is bigger than triangle **T f b** which is bigger than segment **T f y b**, which is outside the circle and it is the same with the others.	فمثلث ق ف ط اعظم من مثلث ط ف ب الذي هو اعظم من قطعة ط ف ي ب الخارجة من الدائرة وكذلك في الباقي[46]
Let the spaces left between the circle and the	The four triangles at the angular points of the	والمثلثات[47] الاربعة التي على زوايا المربع تفصل

[42] H: ينصف
[43] H: مماسا
[44] H: معا
[45] H: فق ط
[46] H: البواقى

circumscribed polygon, such as the figure **ΠΖΑ**, be less than the excess by which **E** exceeds the circle **ΑΒΓΔ**. Therefore the circumscribed rectilineal figure is now less than **E**; which is absurd; for it is greater, because **ΝΑ** is equal to the perpendicular of the triangle, while the perimeter is greater than the base of the triangle.

square separate a bigger [part] than half of the remainder of the square after subtraction of the circle. We divide the arcs exactly so, repeatedly, and we draw tangent lines to the circle until the remaining segments outside the circle together are smaller than the excess of the triangle **e** over the circle. So the polygon that lies on the circle is smaller than the triangle **e**.

من باقي المربع بعد نقصان الدائرة منه اعظم من النصف وننصف[48] القسى هكذا مرة بعد اخرى وتخرج الخطوط المماسة للدائرة الى ان تبقى قطع خارجة من الدائرة مجموعها اصغر من زيادة مثلث ه على الدائرة فيكون الشكل الكثير الاضلاع الذي على الدائرة اصغر من مثلث ه

But the rectangle contained by **n f**, the radius, and the perimeter of the figure that circumscribes the circle, I mean, twice the magnitude of the figure is greater than twice the triangle, because of the fact that the perimeter of the figure is greater than the perimeter of the circle. The figure is bigger than the triangle but it was smaller than it. That is a contradiction.

ولكن[49] سطح ن ف نصف القطر في محيط الشكل الذي على الدائرة اعني ضعف مقدار الشكل اعظم من ضعف المثلث لكون محيط الشكل اعظم من محيط الدائرة فالشكل اعظم من المثلث وكان اصغر منه هذا خلف

The circle is therefore equal to the triangle **E**.

So the circle is equal to the triangle **e**, so the rectangle contained by the radius and half of the perimeter [of the circle] is equal to the surface of the circle and that it what we wanted.

فاذا[50] الدائرة مساوية بمثلث ه فسطح نصف القطر في نصف المحيط مساو لسطح الدائرة وذلك ما اردناه

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47 H: فالمثلثات

48 H: تنصف

49 T: ليكن

50 H: اذا

[In addition] it has been demonstrated with this [proof] that the rectangle contained by the radius and half of a part of the perimeter is equal to the circular sector that is contained by that part and the two lines issuing from the centre to the two endpoints of that part.

وقد بان من ذلك ايضا ان سطح نصف القطر في نصف قطعة من المحيط يكون مساوياً[51] للقطاع الذي محيط به تلك القطعة مع الخطين الخارجين من المركز الى طرفي تلك القطعة

The circumference of any circle is greater than three times the diameter and exceeds it by a quantity less than the seventh part of the diameter but greater than the seventy-first parts.

2) The perimeter of the circle is longer than three times its diameter by [a magnitude] less than one-seventh of the diameter and more than $\frac{10}{71}$ of the diameter.

(ب) محيط الدائرة اطول من ثلاثة اضعاف قطرها باقل من سبع القطر واكثر من عشرة اجزاء من احد وسبعين جزءا من القطر

Let there be a circle with diameter **ΑΓ** and centre **E**, and let **ΓΛZ** be a tangent and the angle **ZEΤ** one-third of a right angle.

So let **a g** be the diameter of the circle and **e** its centre and **d z** a tangent to the circle. The angle **z e g** is a third of a right angle. I mean, half of an angle [i.e. of one of the angles] of an equilateral triangle.

فليكن ا ج قطر الدائرة و ه مركزها و د ز مماساً[52] للدائرة وزاوية ز ه ج ثلث زاوية قائمة اعني نصف زاوية من زوايا المثلث المتساوي الاضلاع

Then the ratio **EΓ** to **ΓZ** is greater than the ratio 265 to 153 and the ratio **EZ** to **ZΓ** is equal the ratio 306 to 153.

So the ratio **e z** to **z g** is the ratio 2:1 and let this be as the ratio 306:153. If we subtract the square of the number that is opposite to **z g** from the square of the number that is opposite to **e z**, and we take the root of what is left, **e g** is in this magnitude more than 265 and the difference is

فنسبة ه ز الى ز ج هي نسبة الاثنين الى الواحد ولتكن كنسبة ٣٠٦ الى ١٥٣ واذا القينا[53] مربع العدد الذي بازاء ز ج من مربع العدد الذي بازاء ه ز واخذنا جذر الباقي كان ه ج بذلك المقدار اكثر

[51] H: مساويا
[52] H: مماسا
[53] H: الفنيا

	some fraction [i.e. is less than one].	من ٢٦٥ بكسر ما
Now let the angle **ZEΓ** be bisected by **EH**. It follows that the ratio **ZE** to **EΓ** is equal to the ratio **ZH** to **HΓ**	We bisect the angle **z e g** at **h** by a line **e h**, then the ratio **z e** to **e g** is as the ratio **z h** to **h g**.	وننصف زاوية زهج على ح بخط هح فنسبة زه الى هج كسنبة زح الى حج
so that the ratio **ZE**+**EΓ** to **ZΓ** is equal to the ratio **EΓ** to **HΓ**. Therefore the ratio **ΓE** to **ΓH** is greater than the ratio 571 to 153,	And *compenendo* and *seperando* the ratio **z e** and **e g** combined to **z g** is as the ratio **e g** to **g h**. If we add up the numbers that belong to **z e** [and] **e g**, [then] that is more than 571. So we make it opposite to **e g** and what is opposite at **g h** becomes in this measure 153.	واذا ركبنا[54] وابدلنا كانت نسبة زه هج معا الى زج كنسبة هج الى جح فاذا جمعنا العددين اللذين بازاء زه هج كان اكثر من ٥٧١ فنجعله بازاء هج ويصير الذي بازاء جح بهذا المقدار ١٥٣
hence the ratio **EH**^{2} to **HΓ**^{2} is greater than the ratio 349450 to 23409, so that the ratio **EH** to **HΓ** is greater than the ratio $591\frac{1}{8}$ to 153.	If we add their squares and we take the root of that, [then] **e h** in this measure is greater than $591\frac{1}{8}$.	واذا جمعنا مربعيهما واخذنا جذرهما كان هح بهذا المقدار اكثر من ٥٩١ وثمن
Again, let the angle **HEΓ** be bisected by **EΘ**; then by the same reasoning The ratio **EΓ** to **ΓΘ** is greater than the ratio $1162\frac{1}{8}$ to 153, so that the ratio **ΘE** to **ΘΓ** is greater than the ratio	And again we bisect the angle **h e g** at **T** by a line **e T**. As has been done before; the ratio **h e** [plus] **e g** to **h g** is as the ratio **e g** to **g T**. If we add up the numbers [that belong to] **h e** and **e g** and we place the two opposite to **e g**, [then] **e g** is more than $1162\frac{1}{8}$ and **T g** is in this measure 153. As has been explained before **e T** is in this measure more than	وايضا ننصف زاوية حهج على ط بخط هط ويكون كما تقدم نسبة حه هج الى حج كنسبة هج الى جط واذا جمعنا عددي حه هج وجعلناهما بازاء هج كان هج اكثر من ١١٦٢ وثمن وطج بذلك المقدار ١٥٣ ويكون بمثل ما مر هط بذلك المقدار اكثر من ١١٧٢ وثمن

[54] H: واذار كبنا

$1172\frac{1}{8}$ to 153.

Again, let **ΘΕΓ** be bisected by **ΕΚ**. Then the ratio **ΕΓ** to **ΓΚ** is greater than the ratio 2334¼ to 153,

so that the ratio **ΕΚ** to **ΓΚ** is greater than the ratio 2339¼ to 153.

$1172\frac{1}{8}$.

We bisect also the angle **T e g** at **k** by the line **e k**. The ratio **T e** [plus] **e g** to **T g** is as the ratio **e g** to **g k**. This change makes [the number that] is opposite to **e g** more than $2334\frac{3}{8}$ and [the number] that is opposite to **g k** [becomes] 153. **e k** is in this measure more than $2339\frac{3}{8}$.

وننصف ايضا زاوية ط ه ج على ك بخط ه ك وتكون نسبة ط ه ه ج الى ط ج كنسبة ه ج الى [55] ج ك فتصير هذه النوبة بازاء ه ج اكثر من ٢٣٣٤ وربع وثمن وبازاء ج ك ١٥٣ ويكون ه ك بهذا المقدار اكثر من ٢٣٣٩ وربع وثمن

Again, let the angle **ΚΕΓ** be bisected by **ΛΕ**. Then the ratio **ΕΓ** to **ΛΓ** is greater than the ratio 4673½ to 153.

We bisect also the angle **k e g** at **l** by the line **e l**. By the above-mentioned analogy, [the number] that is opposite to **e g** becomes more than 4673¾, **g l** is in this measure 153.

وننصف ايضا زاوية ك ه ج على ل بخط ه ل ويصير على القياس المذكور بازاء ه ج اكثر من ٤٢٧٣ ونصف وربع ويكون ج ل بهذا المقدار ١٥٣

Now since the angle **ΖΕΓ**, which is the third part of a right angle, has been bisected four times, the angle **ΛΕΓ** is $\frac{1}{48}$ of a right angle. Let the angle **ΓΕΜ** be placed at **Ε** equal to it. The angle **ΛΕΜ** is therefore $\frac{1}{24}$ of a right angle. And **ΛΜ** is therefore the side of a polygon escribed to the circle and having ninety-six sides.

Because of the angle **z e g** being a third of a right [angle], the angle **l e g** is $\frac{1}{48}$ of a right [angle]. We construct on the point **e** of the line **g e** the angle **g e m** equal to the angle **g e l**, then the angle **l e m** is $\frac{1}{24}$ of a right [angle], and the side **l m** is a side of a regular figure consisting of 96 sides which circumscribe the circle.

فلكون زاوية ز ه ج ثلث قائمة تكون زاوية ل ه ج جزءا من ثمانية واربعين جزءا من قائمة ونعمل على نقطة ه من خط ج ه زاوية ج ه م مثل زاوية ج ه ل فزاوية ل ه م جزء من اربعة وعشرين جزءا من قائمة ويكون ضلع ل م ضلع الشكل المتساوي الاضلاع والزوايا ذي الستة والتسعين ضلعا المحيط بالدائرة

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[55] H: adds الى خط

Since the ratio **ΕΓ** to **ΓΛ** was proved to be greater the ratio 4673½ to 153 and **ΑΓ** is equal to 2**ΕΓ**, **ΛΜ** is equal to 2**ΓΛ**, the ratio of **ΑΓ** to the perimeter of the 96-sided polygon is greater than the ratio 4673½ to 14688. And the ratio is greater than 3,

being in excess by 667½, which is less than the seventh part of 4673½; so that the escribed polygon is greater than three times the diameter by less than the seventh part;

Now if we multiply the number that is opposite to **l m** 96 times, the product of this number reaches 14688. The diameter is in this measure 4673½, so that [number] which is opposite to the perimeter of the figure is more than 3 times that [number] which is opposite to the diameter with [an excess of] 667½, to which the ratio to the number of the diameter is less than $\frac{1}{7}$. So the perimeter of the [above-]mentioned figure is longer than 3 times the diameter of the circle by less than $\frac{1}{7}$ times the diameter.

فاذا ضربنا العدد الذي بازاء ل م في ستة وتسعين بلغ ضعف هذا العدد ١٤٦٨٨[56] ويكون القطر بذلك المقدار ضعف ٤٦٧٣ ونصف فالذي[57] بازاء محيط الشكل اعظم من ثلاثة امثال الذي بازاء القطر بست مائة وسبعة وستين ونصف التي نسبتها الى عدد القطر اقل من السبع فاذا محيط الشكل المذكور اطول من ثلاثة امثال قطر دائرة بانقص من سبع القطر

a fortiori therefore the circumference of the circle is less than $3\frac{1}{7}$ times the diameter.

And the difference between the circumference of the circle and $3\frac{1}{7}$ times the diameter is greater than that difference. Necessarily.

و يكون نقصان محيط الدائرة من ثلاثة امثال القطر وسبعة اكثر من ذلك النقصان لامحالة

Let there be a circle with diameter **ΑΓ** and the angle **ΒΑΓ** one-third of a right angle.

Then the ratio **ΑΒ** to **ΒΓ** is less than the ratio 1351 to 780.

We repeat the circle. Its diameter is **a g** and we draw on it the angle **g a b**, a third of a right [angle]. Let the ratio **a g** to **g b**, which is 2:1, be as the ratio 1560:780. Then **a b** is in this measure less than 1351.

ونعيد الدائرة على قطرها ا ج ونرسم عليه زاوية ج ا ب ثلث قائمة ولتكن نسبة ا ج الى ج ب التي هي نسبة الاثنين الى الواحد كنسبة ١٥٦٠ الى ٧٨٠ فيكون ا ب بذلك المقدار اقل من ١٣٥١

[56] H: ١٤٤٨٨

[57] H: فالسذى

Let **ΒΑΓ** be bisected by **ΑΗ**. Now since the angle **ΒΑΗ** is equal to the angle **ΗΓΒ** and the angle **ΒΑΗ** is equal to the angle **ΗΑΓ**, therefore the angle **ΗΓΒ** is equal to the angle **ΗΑΓ**. And the right angle **ΑΗΓ** is common. Therefore the third angle **ΗΖΓ** is equal to the third angle **ΑΓΗ**.

We bisect the angle **b a g** by the line **a h** and we connect **g h**. So in the triangles **a h g**, **g h z** and **a b z**, the angles **h a g**, **h g z** and **b a z** are equal and the two angles [in] **h** [and] **b** are right.

وننصف زاوية[58] ب ا ج بخط ا ح ونصل ج ح ولان في مثلثات ا ح ج ج ح ز ز ا ب[59] زوايا ح ا ج ح ج ز ب ا ز متساوية وزاويتا[60] ح ب قائمة

The triangle **ΑΗΓ** is therefore equiangular with the triangle **ΓΗΖ**; therefore the ratio **ΑΗ** to **ΗΓ** is equal to the ratio **ΓΗ** to **ΗΖ** which is equal to the ratio **ΑΓ** to **ΓΖ**. But the ratio **ΑΓ** to **ΓΖ** is equal to the ratio **ΓΑ**+**ΑΒ** to **ΒΓ**. Therefore the ratio **ΒΑ**+**ΑΓ** to **ΒΓ** is equal to the ratio **ΑΗ** to **ΗΓ**. Therefore the ratio **ΑΗ** to **ΗΓ** is less than the ratio 2911 to 780. Hence the ratio **ΑΓ**^{2} to **ΓΗ**^{2} is equal to the ratio **ΑΗ**^{2}+**ΗΓ**^{2} to **ΓΗ**^{2} which is less than the ratio 2911^2+780^2 to 780^2, so that the ratio **ΑΓ** to **ΓΗ** is

[Then] the triangles are similar and because of that the ratio **a h** to **h g** is as the ratio **h g** to **h z** and as the ratio **a g** to **g z** and as the ratio **a b** to **b z**. Indeed, [it] is as the ratio of **g a** and **a b** combined to **g b**. The ratio of **g a** and **a b** combined to **g b** is as the ratio of **a h** to **h g** and the number of **a g** and **a b** combined is less than 2911 and the number of **g b** is 780.
So if we add up the two [numbers] which are opposite to **a h** and **e g**, then **a g** is in this measure

فتكون[61] المثلثات متشابهة وتكون لذلك نسبة ا ح الى ح ج كنسبة ج ح[62] الى ح ز وكنسبة ا ج الى ج ز وكنسبة ا ب الى ب ز بل كنسبة ج ا ا ب جميعاً[63] الى ج ب ونسبة ج ا ا ب جميعاً[64] الى ج ب كنسبة ا ح الى ح ج وعددا ج ا[65] ا ب جميعاً[66] اقل من ٢٩١١ وعدد ج ب ٧٨٠ فاذا جعلناهما بازاء ا ح ح ج كان ا ج بذلك المقدار اقل من ٣٠١٣[67] ونصف وربع

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[58] H: زواية
[59] H: ا ب ز
[60] H: متساويةو زوايا
[61] H: تكون
[62] H: ح ج
[63] H: جميعا
[64] H: جميعا
[65] H: ا ج
[66] H: جميعا
[67] T: ٣١٠٣

less than the ratio 3013¾ to 780.

Let the angle **ΓΑΗ** be bisected by **ΑΘ**.

By the same reasoning the ratio **ΑΘ** to **ΘΓ** is less than the ratio 5924¾ to 780,

so that is less than the ratio $\frac{4}{13}$ times 5924¾ to $\frac{4}{13}$ times 780, so that is less than the ratio 1823 to 240.

Therefore the ratio **ΑΓ** to **ΓΘ** is less than the ratio $1838\frac{9}{11}$ to 240.

Further, let the angle **ΘΑΓ** be bisected by **ΚΑ**.

Then the ratio **ΑΚ** to **ΚΓ** is less than the ratio $\frac{11}{40}$ times $3661\frac{9}{11}$ to $\frac{11}{40}$ times 240, so that is less than the ratio 1007 to 66. Therefore the ratio **ΑΓ** to **ΚΓ** is less than the ratio $1009\frac{1}{6}$ to 66.

Further, let the angle **ΚΑΓ** be bisected by **ΛΑ**.

Then the ratio **ΑΛ** to **ΛΓ** is less than the ratio $2016\frac{1}{6}$ to 66.

Therefore the ratio **ΑΓ** to **ΓΛ** is less than the ratio

less than 3013¾.

We bisect the angle **h a g** by the line **a T** and we connect **T g**. According to the above-mentioned analogy is [the number] which is opposite to **a T** less than 5924 and [the number] opposite to **T g** is 780. That is in the ratio 1823:240, because the ratio of each of the first numbers [i.e. 5924 and 780] to the corresponding numbers [i.e. 1823 and 240] is the ratio of 3¼:1. **a g** is in this measure less than $1838\frac{9}{11}$, corresponding to the one [i.e. after division by 3¼].

We bisect the angle **T a g** by the line **a k**, then [the number] which is opposite to **a k** is less than $3661\frac{9}{11}$ and [the number] opposite to **k g** is 240, that is in the ratio of 1007:66. That is because the ratio of each of the two [i.e. $3661\frac{9}{11}$ and 240] to the corresponding of the two [i.e. 1007 and 66] is the ratio 40:11.

We bisect the angle **l a g** by the line **a l**, then [the number] which is opposite to **a l** is greater than $2016\frac{1}{6}$ and [the number] opposite to **l g** is 66 and **a g** is in this measure 2017¼.

So the ratio **a g** to **g l** is less than the ratio

وننصف زاوية ح ا ج بخط ا ط ونصل ط ج فيكون على قياس ما مر بازاء ا ط اقل من ٥٩٢٤ وبازاء ط ج ٧٨٠ ويكون ذلك على نسبة ١٨٢٣ الى ٢٤٠ لأن نسبة كل واحد من العددين الاولين الى نطيره من هذين العددين نسبة ثلاثة وربع الى واحد ويكون ا ج بهذا المقدار اقل من ١٨٣٨ وتسعة اجزاء من احد عشر جزءا من الواحد

وننصف زاوية ط ا ج بخط ا ك فيكون بازاء ا ك اصغر من ٣٦٦١ وتسعة اجزاء الى احد عشر وبازاء ك ج ٢٤٠ ويكون على نسبة ١٠٠٧ الى ٦٦ لأن نسبة كل واحد منهما الى نطيره من هذين نسبة اربعين الى احد عشر

وننصف زاوية ك ا ج بخط ا ل فيكون بازاء ا ل اقل من ٢٠١٦ وسدس وبازاء ل ج ٦٦ ويكون ا ج بذلك المقدار ٢٠١٧ وربع فنسبة ا ج الى ج ل اصغر من نسبة ٢٠١٧ وربع الى ٦٦ واذا ضربنا

2017¼ to 66

and *invertendo* the perimeter of the polygon bears to the diameter a ratio greater than the ratio 6336 to 2017¼, which is greater than $3\frac{10}{71}$.

Therefore the perimeter of the 96-sided polygon is greater than $3\frac{10}{71}$ times the diameter, so that *a fortiori* the circle is greater than $3\frac{10}{71}$ times the diameter.
The perimeter of the circle is therefore more than three times the diameter, exceeding by a quantity less than $\frac{1}{7}$ part but greater than $\frac{10}{71}$ parts.

2017¼:66. If we multiply 66 by 96, then all the sides of the polygon of 96 sides inscribed in the circle is 6336. That is more than 3 times 2017¼ with a remainder of $\frac{10}{71}$ of one [i.e. $\frac{10}{71}$ of 2017¼].

So the perimeter of the [above-]mentioned regular figure that is inscribed the circle is greater than 3 times its diameter with a remainder of $\frac{10}{71}$ [times its diameter]. The perimeter of the circle is greater than that so the perimeter of the circle is greater than 3 times its diameter by less than $\frac{1}{7}$ [times its diameter] and more than $\frac{10}{71}$ [times its diameter] and that is what we wanted.

I say that there is another method of the astronomers and that is that they obtain a chord of a small arc which is an integer part of the

ستة وستين في ستة وتسعين صار جميع اضلاع الشكل ذي الستة والتسعين ضلعا الذي على الدائرة ٦٣٣٦ وهو اكثر من ثلاثة اضعاف الفين وسبعة عشر وربع باكثر من عشرة اجزاء من احد وسبعين جزءاً[68] من واحد

فمحيط الشكل المتساوي الاضلاع والزوايا المذكور الذي[69] على الدائرة تزيد على ثلاثة اضعاف قطرها باكثر من عشرة اجزاء من احد وسبعين جزءا من واحد ومحيط الدائرة اعظم منه فاذن[70] محيط الدائرة يزيد على ثلاثة اضعاف قطرها باقل من سبعة واكثر من عشرة اجزاء الى احد وسبعين جزءا وذلك ما اردناه

اقول وللمنجمين طريق آخر وهو انهم يحصلون وتر قوس صغيرة يكون جزءا من

[68] H: جزءا
[69] H: المذكورة التي
[70] H: فاذا

perimeter with the fundamental principles that have been explained in the *Almagest* and other books of them containing demonstrations.

They make it one side of a figure inscribed in the circle. Its ratio to the perpendicular [line] from the centre of the circle to it [i.e. the created side] is as the ratio of the side of the similar figure that is circumscribed around the circle to half of the diameter.

So they also deduce this side. Then they obtain, by the computation of these two [quantities] the two quantities, such that the perimeter is greater than one of them and less than the other. So the perimeter is obtained with a close approximation.

The example of that is: let the circle be **a b**, its centre is **g** and **a b** is $\frac{1}{720}$ part of the perimeter. We connect the chord **a b**; then its measure, according to a calculation of Abū al-Wafā al-Būzjānī according to the [above-]mentioned principles with a very close approximation, is 0; 31, 24, 55, 54, 55. That is a chord of half a degree,

محيط الدائرة بالاصول التي تبينت في كتاب المجسطي وغيره من كتبهم البرهانية

ويجعلونه ضلعا من اضلاع الشكل الذي في الدائرة وتكون نسبته الى العمود الواقع من مركز الدائرة عليه كنسبة ضلع الشكل الذي على الدائرة الشبيهة به الى نصف القطر

فيحصلون ذلك الضلع ايضا يحصلون بحسبهما المقدارين الذين يزيد المحيط على احدهما وينقص من احدهما فيتحصل المحيط باقرب تقريب

مثاله لتكن الدائرة ا ب ومركزها ج و ا ب منه جزء من سبع مائة وعشرين جزءا من[71] المحيط ونصل وتر ا ب فيكون مقداره بحساب ابي الوفا البوزجاني على الاصول المذكورة باقرب تقريب ه لا كد نه ند نه

[71] H en T: هي, scribal error

خامسة وهو وتر نصف درجة اذا جعل القطر
مائة وعشرين جزءا

if the diameter is made 120 parts.

واذا جعلناه ضلع شكل ذي سبع مائة
وعشرين ضلعا في الدائرة يكون محيط ذلك
الشكل بحسبه ٣٧٦ نط ي نط ثالثة واذا
نصفنا وتر نصف درجة كان مقدار ا د هو[72]
ه يه مب كز نز كز خامسة و[73] مربعه ه د و
مد ب د نز كه يح[74] ل ط عاشرة

And if we make it [i.e. the chord] as a side of a regular figure that has 720 sides inside the circle, then the perimeter of that figure is according to his calculation 376; 59, 10, 59. If we bisect the chord of half a degree [then] the measure of **a d** is 0; 15, 42, 27, 57, 27 and the square of it is 0; 4, 6, 44, 2, 4, 57, 25, 18, 30, 9.

ومربع نصف القطر الذي هو خط ا ج ٣٦٠٠
جزءا نقصنا من مربعة ا د منه بقي مربعة د ج
٣٥٩٩ نه نج[75] يه[76] نز نه ب لد[77] ما كط[78]

The square of the radius, that is the line **a g**, is 3600 parts. We subtract the square of **a d** of it [so that] the square of **d g** remains, which is 3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51.

[72] H: omitted
[73] H: omitted
[74] T: لح
[75] H: كج, T: ambiguous
[76] H: نه
[77] T: possibly لو
[78] H: omitted

The square root of it is the line **d g**, which is 59; 59, 57, 56, 37, 56, 51.

We multiply **a d** with **g h**, the radius, and we divide it by **d g**. The measure of **h e** remains and is 0; 15, 42, 28, 29, 45.

We double it and it becomes 0; 31, 24, 56, 59, 31. which is the measure of **e z**, which is a side of a figure that has 720 sides circumscribed around a circle (that is equal to the circle before).

The perimeter of the figure is after calculation 376: 16, 59, 23, 54, 12. So if we make the diameter 120, the perimeter [i.e. of the circle] is 376 parts and a fraction that is greater than 0; 59, 10, 59, 0 and less than 0; 59, 23, 54, 12, and if we change the two to the measure that Archimedes

نا[79] جذره هو خط د ج نط نط نز نو[80] لز[81] نو نا سادسة

ضربنا ا د في ج ح نصف القطر وقسمناه على د ج خرج مقدار[82] ح ه[83] ه[84] يه مب كح[85] كط مه خامسة

ضعفناه بلغ ه لا كد نو[86] نط لا خامسة وهو مقدار ه ز وهو ضلع شكل ذي سبع مائة وعشرين ضلعا على الدائرة شبيهة بالاول

ومحيط الشكل بحسبه يكون ٣٧٦ يو[87] نط كج ند يب خامسة ايضا فاذا جعلنا القطر مائه وعشرين كان المحيط ٣٧٦ جزءا وكسرا اكثر من نط ي نط ه رابعة واقل من نط كج

[79] H: omitted
[80] H: omitted
[81] H: omitted
[82] H: مقداره
[83] T: ه ح
[84] H: omitted
[85] T: omitted
[86] H: نز
[87] H: يو

mentioned, the perimeter is greater than 3 times the diameter plus something that is greater than $\frac{10}{70;38,41,21}$, and less than $\frac{10}{70;37,47,37}$, which is approximately $\frac{10}{70;38,14,29}$.

ند يب رابعة واذا حولناهما الى المقدار الذي ذكره ارشميدس كان المحيط يزيد على ثلاثة امثال القطر بما هو اكثر من عشرة اجزاء من سبعين جزءا ولح ما كا ثلاثة واقل من عشرة اجزاء من سبعين جزءا ولز[88] مز لز[89] ثلاثة ويكون بالتقريب عشرة اجزاء من سبعين جزءا ولح يد[90] كط ثلاثة

The circle bears to the square on the diameter the ratio 11 to 14.

3) If the perimeter of the circle is $3\frac{1}{7}$ times the diameter, and this is an approximate ratio used by the surveyors then the ratio of the surface area of the circle to the square of its diameter is [the] ratio 11 to 14 according to this computation.

اذا كان محيط الدائرة ثلاثة امثال القطر وسبعه وهي نسبة تقريبة اصطلح[91] المساحون كانت نسبة سطح الدائرة الى مربع قطرها نسبة احد عشر الى اربعة عشر بحسب ذلك

Let there be a circle with diameter **AB**, and let the square **ΓH** be circumscribed, and let **ΔE** be equal to two times **ΓΔ**, **EZ** equal to $\frac{1}{7}$ times **ΓΔ**. Then, since the ratio **AΓE** to **AΓΔ** is the ratio 21 to 7, while the ratio **AΓΔ** to **AEZ** is the ratio 7 to 1, it

So let the diameter of the circle be **a b** and draw around it the square **g h**. Let **g d** be half of **d e** and **h z** one-seventh of **g d**. Because the ratio of the triangle **a g e** to the triangle **a g d** is the ratio of 21 to 7, and the ratio of the triangle **a g d** to the

وليكن قطر الدائرة ا ب ونرسم عليه مربع ج ح وليكن ج د نصف د ه وه ز سبع ج د فلأن نسبة مثلث ا ج ه الى مثلث ا ج د

[88] T: نز
[89] H: كز
[90] T: possibly ند
[91] H: adds اصطلح عليه

follows that the ratio **ΑΓΖ** to **ΑΓΔ** is the ratio 22 to 7.

triangle **a h z** is the ratio of 7 to 1, the ratio of the triangle **a g z** to the triangle **a g d** is the ratio of 22 to 7.

نسبة احد وعشرين الى سبعة ونسبة مثلث ا ج د الى مثلث ا ه ز نسبة سبعة الى واحد تكون نسبة مثلث ا ج ز الى مثلث ا ج د نسبة اثنين وعشرين الى سبعة

But the square **ΓΗ** is 4 times **ΑΓΔ**, while the triangle **ΑΓΔΖ** is equal to the circle **ΑΒ**;

The square of **g h** is 4 times the triangle **a g d**, and the triangle **a g z** is equal to the surface area of the circle so **a g** is equal to half of the diameter and **g z** is approximately equal to the [i.e. the circle's] perimeter.

ومربع ج ح اربعة امثال مثلث ا ج د ومثلث ا ج ز مساو لسطح الدائرة لان ا ج مساو لنصف القطر وج ز مساو بالتقريب للمحيط

therefore the circle bears to the square **ΓΗ** the ratio 11 to 14.

So the ratio of the square of the diameter to the surface area of the circle is [as] the ratio of 28 to 22, that is, as the ratio of 14 to 11 and that is what we wanted.

فنسبة مربع القطر الى سطح الدائرة نسبة ثمانية وعشرين الى اثنين وعشرين بل نسبة اربعة عشر الى احد عشر وذلك ما اردناه

This is the completion of the account on the measurement of the circle. Let us finish the treatise by praising God, the most exalted; the good result is because of Him.

وهذا اتمام القول في تكسير الدائرة ولنقطع الكلام حامدين للة تعالى على حسن توفيقه

Figures from al-Ṭūsī's text

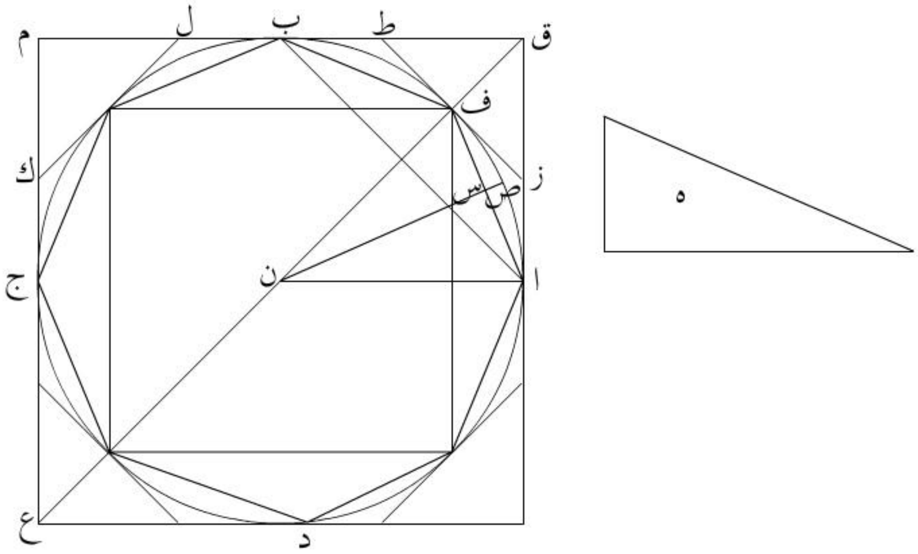

Figure 1: al-Ṭūsī's first proposition.

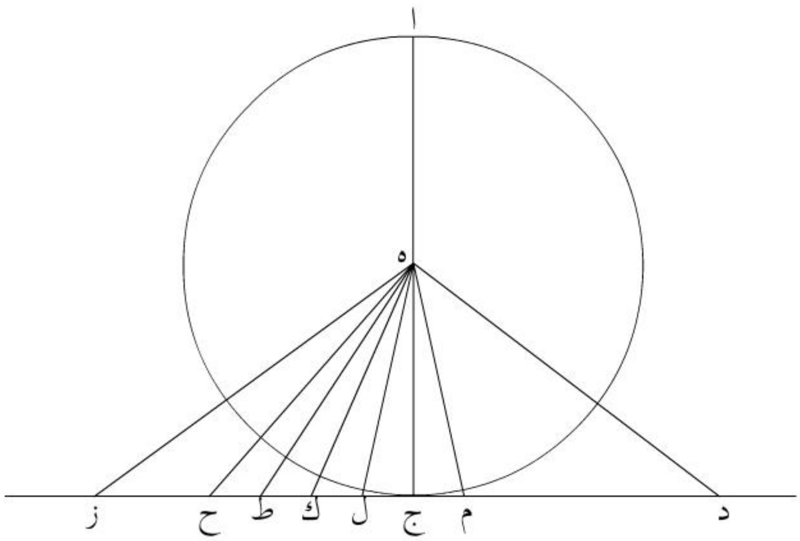

Figure 2: al-Ṭūsī's second proposition, circumscription.

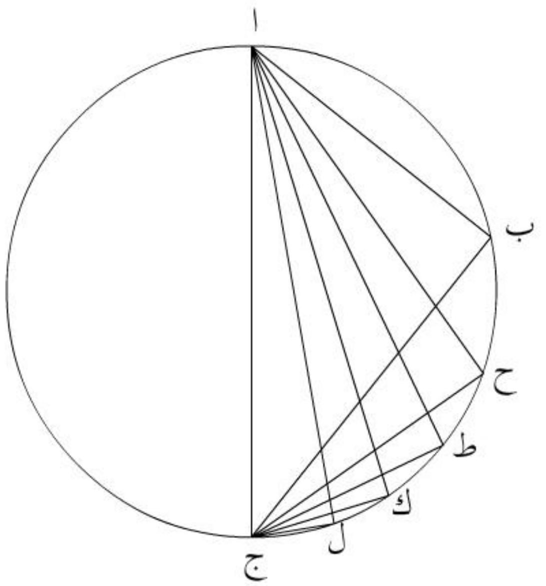

Figure 3: al-Ṭūsī's second proposition, inscription.

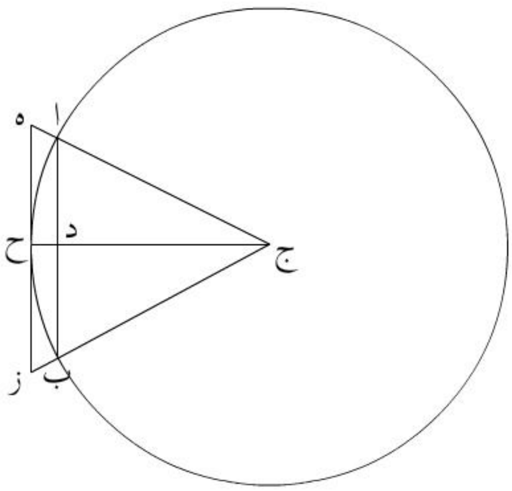

Figure 4: al-Ṭūsī's second proposition, another proof from the astronomers.

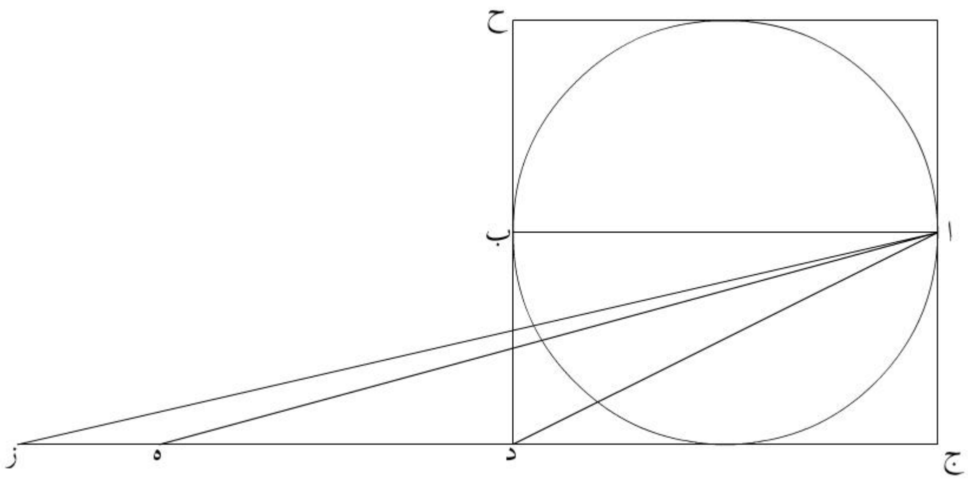

Figure 5: al-Ṭūsī's third proposition.

Chapter 3

The argument of the first proposition

In the first proposition, it is proved that the surface area of a circle is equal to the surface area of a right triangle of which one of the two right-angled sides is equal to the radius and the other the perimeter. Using modern formulas it is easy to see that this is correct: the surface of a triangle is $\frac{1}{2} \times h \times w$ with h = height and w = width. In a right triangle the two right-angled sides are the height and the width and in this special case they are r (the radius) and $2\pi r$ (the perimeter) so the surface area becomes $\frac{1}{2} \times r \times 2\pi r = \pi r^2$ and as we know that is exactly the surface of a circle. Archimedes' reasoning is actually a proof for these formulas.

His proof consists of investigating what happens if the circle would be either greater or less than the triangle. In both cases a regular polygon can be constructed such that its surface area is both bigger and smaller, resulting in a contradiction (and so, via *reductio ad absurdum*, come to the conclusion that the surface area of the circle must be equal to the surface area of the triangle). A brief elaboration on the hypothesis if the circle were bigger should suffice to understand the process, a similar process could be constructed for the hypothesis that the circle is smaller than the triangle. First assume the circle to be greater than the triangle. An inscribed regular polygon is constructed by repeatedly bisecting the chords and connecting the points. See figure 1 (and figure 6); if you draw a square inside the circle and outside the circle, the points at which the inside-square touches the circle can be connected with the points at which the outside-square touches the circle, and a regular octagon is constructed. This process can be repeated of course to get a regular 16-sided polygon, 32-sided polygon etc. The trick is, if you repeat this process the figure can come indefinitely close to the circle, making sure that it leaves not even a enough room for the circle to be equal to the (supposed) excess of the circle over the triangle (i.e. the regular polygon can be made such that it is bigger than the triangle). That this is the process that should be followed is not explicitly stated by Archimedes who merely states the construction of the inscribed square and the bisection of the arcs (p. 16-17). That this process, if repeated, can separate any small amount you like between the circle and the regular polygon is explicitly proved in Euclid's *Elements* XII.2. Al-Ṭūsī states this proof saying that 'the resulting triangles separate more than half of the sections' adding that this has been proved before (it seems he is referring to Euclid's proof but it could of course also be that he proved this himself in another text). Al-Ṭūsī also states that this should be repeated until the result is that segments between the circle and the regular polygon are smaller than the excess of the circle over the triangle. Only with the additions of al-Ṭūsī does the proof become entirely clear and unambiguous.

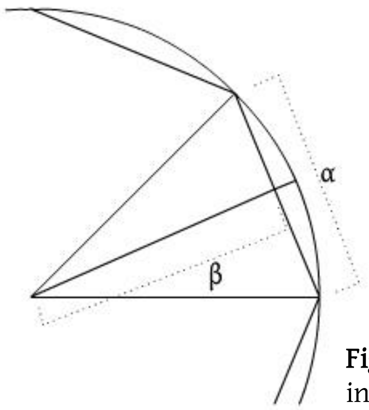

Figure 6: Surface area of the inscribed polygon

So now we know that the polygon is greater than the triangle. What remains to be proven is that it is simultaneously less than the triangle, arriving at a contradiction. In modern notation this is done by comparing the formulas of the surface areas of the regular polygon and the triangle. The surface area of the triangle is of course $\frac{1}{2} \times r \times p$ with r for radius and p for perimeter (of the circle). The surface area of the regular polygon is obtained by adding all the surface areas of the triangles that are made (i.e. multiplying the surface area of one triangle by n, for an n-sided polygon). So this becomes $n \times \frac{1}{2} \times \alpha \times \beta$ with α the width of a side and β the height of a triangle (see Figure 6). From the figure you immediately see that $\beta < r$ and $n \times \alpha < p$. So $\frac{1}{2} \times \beta \times n \times \alpha < \frac{1}{2} \times r \times p$, so the surface area of the polygon is less than the surface area of the triangle. Here it is again interesting to compare Archimedes and al-Ṭūsī. Archimedes merely states that the altitude (**NΞ**) is less than the side of the triangle (meaning, the side of **E** which is equal to the radius of the circle). He adds that the perimeter of the regular polygon is less than the triangle **E** (meaning, less than the side of **E** which is equal to the perimeter of the circle). From this he concludes without further ado that this is absurd (p. 17-18). This is of course not enough to be called a complete proof and al-Ṭūsī felt probably likewise uncomfortable with it as he tried to fill in the gaps. He makes the step from unequal sides to unequal surface areas. He first states that the rectangle contained by **n s** [the altitude] times the perimeter of the figure is twice the surface area of the regular polygon (which can easily seen to be correct using modern notation) and then states that this is smaller than twice the surface area of the triangle (i.e. the rectangle contained by the radius and the perimeter of the circle). From this he then concludes that the surface area of the regular polygon is smaller than the triangle. The polygon can't be greater *and* less at the same time, so we have to drop the assumption that the circle is bigger than the triangle. The proof works likewise if the circle were to be imagined to be less than the triangle. Now the polygon is circumscribed instead of inscribed.

Already in the first paragraph (p. 16) al-Ṭūsī makes an elaboration, where he states that the result is that a circle is equal to a rectangle (*saṭḥ*) with sides equal to half of the circle's diameter and half of its perimeter. This seems a comment to make the proposition more precise and he uses language like this frequently (although not always). A bigger task al-Ṭūsī undertakes is the clarification of some of the more vague statements Archimedes makes. For example, Archimedes simply says "let the segments be less than the excess by which the circle exceeds the triangle", while al-Ṭūsī tries to describe the process that one has to do in order to get segments that are indeed less than the excess (a process described in Euclid's *Elements*, XII.2). He does the same when Archimedes merely states that "the perimeter of the rectilinear figure [i.e. the inscribed regular polygon] is therefore less than the triangle **E**; which is absurd." Here al-Ṭūsī shows that therefore also the surface area of the regular polygon is smaller. In the second case, where the circle is smaller than the triangle, again al-Ṭūsī elaborates on the statements Archimedes makes, completing the proof.

Additionally, al-Ṭūsī states that this also proves that the surface of a sector is equal to "the rectangle contained by the radius and half of a part of the perimeter". He comes to the idea of this exactly because he defines the surface as a rectangle. First he noted that the surface of a circle is equal to a rectangle contained by the radius and half of the perimeter, then he probably figured that the ratio of the circle to the sector is equal to the ratio of the perimeter to the segment of the perimeter than defines the sector. In modern notation this amounts to $V(sector) = \frac{s}{2\pi r} \times \pi r^2 = \frac{1}{2} sr$, with $V(sector)$ = the surface area of the sector, s = the amount of degrees that defines the sector and r = the radius of the circle.. This is what al-Ṭūsī states.

The argument of the second proposition

Al-Ṭūsī's second proposition is Archimedes' third proposition. Al-Ṭūsī makes it the second, probably because the third (or, Archimedes' second) proposition relies on the result of this one. He follows Archimedes' reasoning closely and gives no additional information on why the (at first sight) strange numerical ratios are used. In the ratio of **e z**, **z g** and **e g** he does however explain the Pythagorean theorem and even states that **e g** is 265 and a fraction, following Eutocius. Besides that he does not give the full explanation of Eutocius but tends to be more concise.

This proposition gives a lower and upper bound for π with a rather sophisticated proof. Before Archimedes, approximations of π were all in a single rational fraction. Here, Archimedes does not give a single rational fraction for π, but rather computes a lower and upper bound. As before in the first proposition, if you understand one of the approaches, you also understand the other one as it only differs slightly. Let's focus on the upper bound. A regular polygon is constructed circumscribing the circle. The difference with the first proposition is that already at the beginning valuable information on the size of the constructed side of the regular polygon is taken into account. Because the angle is known (30°) the sides can be calculated. As was said before, we do not know exactly how Archimedes came up with such handy approximations of $\sqrt{3}$, and

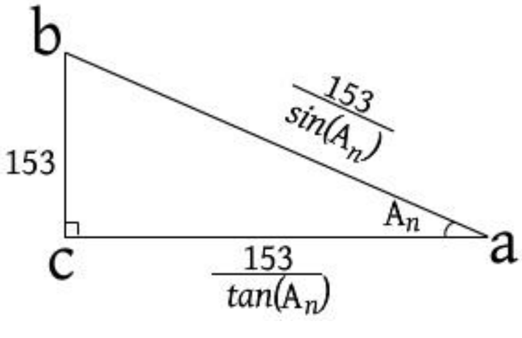

Figure 7: Constructing triangles

likewise we don't know exactly how he made all the following computations. In modern day mathematics it is rather easy to follow. You first start by constructing a triangle with $A_0 = 30°$. Then another triangle is formed with $A_{n+1} = \frac{A_n}{2}$. This can be repeated as many times as desired. In the table below all the numbers are given, with the numbers between brackets not given by Archimedes (nor al-Ṭūsī). The numbers are with regard to Figure 7 in the order **a b**, **a c**, **b c**. It is noteworthy to see that the numbers of **a b** and **a c** come closer and closer to each other. In fact, they can come indefinitely close to each other as in reality only points **a** and **c** are static (they correspond with points **e** and **g** respectively). Point **b** moves closer and closer to **c** (see Figure 2).

	start	*first bisection*	*second*	*third*	*fourth*
e z	306				
e g	265				
z g	153				
e h		$591\frac{1}{8}$			
e g		$(571\frac{1}{7})$			
h g		153			
e T			$1172\frac{1}{8}$		
e g			$(1162\frac{1}{7})$		
T g			153		
e k				2339¼	
e g				$(2334\frac{1}{3})$	
k g				153	
e l					$(4673\frac{2}{3})$
e g					4673½
l g					153

If we formalize the procedure we can come up with the following inequality:

$$s_n = 6 \times 2^n \times \sin A_n < \pi < S_n = 6 \times 2^n \times \tan A_n$$

Where s_n is the perimeter of the inscribed polygon divided by the diameter of the circle, S_n is the perimeter of the circumscribed polygon divided by the diameter of the circle and A_n is the angle needed to make one side of the polygon. A_n is then defined as $A_0 = 30$, $A_n = \frac{A_{n-1}}{2}$. This inequality defines a smaller interval as n increases because the sine and tangent look more and more a like when the angle approaches zero, thus it shows that you can get an estimation of π with an arbitrary small approximation with this method.

In al-Ṭūsī's text, after the third intersection the numbers go wrong. Instead of 2339¼ it reads $2339\frac{3}{8}$, off by $\frac{1}{8}$ (*thumn*). This is then continued through the next numbers. This is not a recalculation by al-Ṭūsī because in the conclusion he returns to the Archimedian value 4673½ for the diameter. Maybe this erroneous *thumn* was added only after al-Ṭūsī but it might as well be there already before al-

Ṭūsī, from the comments al-Ṭūsī makes it is obvious he did not recalculate them. It could be that the scribe was so used to write a $\frac{1}{8}$ after a number (as with most of the numbers before) that he just erroneously continued with it. Another mismatch with the numbers happens when he described the inscribing of the circle. After the second intersection he states the value 5924 where Archimedes states 5924¾. His explanation on why the ratio 5924(¾):780 is equal to 1823:240 is a different approach than Archimedes but is rather obscurely written.

The addition to the second proposition

After the recension of Archimedes' third proposition, al-Ṭūsī begins by stating "I say". This seems to introduce a longer remark, one that is more distant from the original text. Like I said before, Knorr has some mistranslations and the most obvious one is in the first line of the addition to the second proposition, right after "I say". Here, Knorr translates: "I say, and for the two results <there is> another method,..."[92] First of all the 'and' is out of place and is an over precise translation of *wa*, which is in this place rather an intensification of the prefix *li-* following it.[93] Then he translates 'the two results' adding in a footnote "literally: "sources", *manjamaini*, that is, the two bounds just computed."[94] To read this as a dualis of 'source' seems to me a very far-fetched idea and I'm confident that it actually reads *munajjimīni*, being the genitive of the plural of *munajjim*, which means astronomer/astrologer. For one, it suits the sentence better, as it does not need to introduce the '<there is>' Knorr adds. Secondly, it does not need the far-fetched idea that al-Ṭūsī would use 'source' (as in; origin) to refer to the computation. Thirdly, a sentence later there is a reference to 'they' and a little further al-Ṭūsī states that this other proof is backed "with the ground principles that have been clarified in the *Almagest* and other certifying books of them." Again the 'them' is used which can only refer back to the astronomers/astrologer in my opinion, this is also backed by the reference to the *Almagest* (literally: The Great Book), which is of course the book of Ptolemy on astronomy. This would also be a valuable addition in the light of the knowledge level of the reader. As this is a 'middle book', the reader would not have studied the *Almagest* before, but, as anyone wanted to become an astronomer in that time, would be eager to do so. Saying that this addition relies on astronomers and the *Almagest* would certainly raise interest among readers.

And interesting it is indeed. The basic idea is to define a chord that is $\frac{1}{720}$th part of a equilateral figure that is inscribed in the circle. All calculations are now done in the sexagesimal system. The whole calculation is based on a number from a trigonometric table of which the chord of ½° is taken. Al-Ṭūsī takes this number from Abū al-Wafā al-Būzjānī, who created a table with trigonometric numbers. From this number, the rest of the construction is calculated via some simple geometrical

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[92] Knorr, p. 581

[93] People familiar with the Qur'ān should recognize this from high-numbered *suwar*, for example Qur'ān 103:1, "*wa l-ᶜasri*", meaning; "by the declining day!"

[94] Knorr, p. 584

constructions. First the inscribed polygon is properly described, then from the numbers of the inscribed polygon, the size of the outer chord is computed (according to Euclid's *Elements* VI.3). From this number a circumscribed polygon can be constructed and so an upper and lower bound can be established for the circumference of the circle. The conclusion of the proof is similar to the proof of Archimedes, the inner- and outer polygon are compared to the diameter of the circle and in addition the mean of the two values is used as a close approximation of π.

Unfortunately, al-Ṭūsī is already not entirely correct by assuming the chord of ½° to be 0; 31, 24, 55, 54, 55. The correct sexagesimal number is 0; 31, 24, 56, 58, 36, ... and although this is not very far off (by approximately 0,0000046), as Luckey shows the value al-Ṭūsī states is probably mistakenly swapped with the value of Sin(1/2°), which is close to 0; 31, 24, 55, 54, 0.[95] This last value is only $\frac{55}{60^5}$ ($\approx 7 \times 10^{-8}$) off from the value al-Ṭūsī states. So although the error is still very small, it could have been better and this would have definitely improved the concluding approximation of π. Besides this mistake, the numbers are also hard to read sometime and can only be confirmed when we calculate what the numbers should be. For example, the square of **d g** (the square of the apothem) is 3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51. However, if we would follow all the possibilities from the footnotes we could also obtain the value 3599; 55, 23, 55, 57, 55, 2, 36, 41, deviating by more than 0,0083. We know the first number is correct as it should add up to 3600 if we add the square of **a d** (using the Pythagorean theorem, see figure 4, p. 33), so it shows we cannot trust our perception of the text alone.[96] For the apothem of the inscribed polygon, al-Ṭūsī gives 59; 59, 57, 56, 37, 56, 51 which is again really close to the true value of ($60 \times \cos\frac{1}{4} =$) 59; 59, 57, 56, 37, 45,... This time off by approximately 0,000000014. Now al-Ṭūsī states that as the side of the inner-polygon stands to the apothem, the side of the outer-polygon stands to the radius. For al-Ṭūsī this boils down to 0; 31, 24, 56, 59, 31. In reality it is ($120 \times \tan\frac{1}{4} =$) 0; 31, 24, 58, 3,... From these values, the perimeter must be computed and al-Ṭūsī gives 376; 59, 10, 59. If you would multiply al-Ṭūsī's calculated side by 720 you would get 376; 59, 10, 58, 59 while the actual number must be 376; 59, 23, 41,... which is off by approximately 0,0036.[97]

A bigger mistake is made in the computation of the perimeter of the outer-polygon, for which al-Ṭūsī gives 376; 16, 59, 23, 54, 12. A first correction on this number is to neglect the 16 which both Luckey and Knorr do. Actually, just a few lines below this value, al-Ṭūsī states it again but this time

[95] Luckey, P., *Der Lehrbrief über den Kreisumfang (ar-Risāla al-Muḥīṭīya) von Gamšid b. Masᶜūd al-Kāšī übersetzt und erläutert*. Herausgegeben von Alfred Siggel,Akademie Verlag, Berlin, 1953. Reprinted in: Sezgin, F., *Al-Kāsh Texts and Studies*, Islamic Mathematics and Astronomy, vol. 56, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp. 227-329

[96] For example, Knorr gives the erroneous value of 3599; 55, 23, 55, 57, 55, 2, 34, 41. He even gives another erroneous value in a footnote: 3599; 55, 13, 55, 57, 55, 2, 34, 41, 29, 51. He probably did not check his numbers. See Knorr, W.R., p. 582 and p. 584 note 10

[97] See also Knorr, W.R., p. 593

as 376; 59, 23, 54, 12, correcting his earlier mistake. Both Luckey and Knorr do not comment on their emendation although it is quite a big one and I want to elaborate on it as it is also a good opportunity to zoom in on the sexagesimal numbers. As has been shown above, some numbers are ambiguous as they look very similar to other numbers. Examples of this are the *nūn* (ن) and the *yā* (ي) as they would look the same at the beginning of a word (something like ٮ where you would add a dot above or two dots below to decide whether it is an *nūn* or a *yā*). As *yā* means ten and *nūn* means fifty, this is quite a difference. A far less serious ambiguity is the difference between a *wā* (و) and a *zayn* (ز) which represent six and seven respectively. There are more cases where there is ambiguity but these two examples should suffice to prove the point. Another issue is that sometimes it is even not clear which letters belong to a number and which do not. For this, there is an indicator at the end of the number. This is a word that indicates what the power of sixty is for the last number, so it also indicates how many fractions are to be expected. For example *rābiᶜa* ('fourth') is used when there should be four fractions. As these words are very distinct, it is virtually impossible to regard these words as erroneous. This takes us back to the problem of the number 376; 16, 59, 23, 54, 12. In both the Hyderabad edition as the Tehran facsimile edition it clearly states *khāmisa* ('fifth', indicating five fractions) and also Luckey and Knorr confirm this (Luckey even states that also Woepcke has read this).[98] Simply deleting the $\frac{16}{60}$ and stating *rābiᶜa*, although it is correct and is even used this way by al-Ṭūsī a couple of lines later, does not render the text as it was once written by al-Ṭūsī in my opinion. Instead of a scribal error (or whatever assumption Luckey and Knorr made in silence) it can actually be shown to be a small mistake by al-Ṭūsī himself and if we do the calculation of the circumference ourselves it is easily shown. After some calculation al-Ṭūsī gets 0; 31, 24, 56, 59, 31 for one side of a 720-sided polygon that circumscribes the circle. The perimeter of the polygon is then calculated by multiplying this side by 720. So we get $720 \times 0;31,24,56,59,31 = 12 \times 31;24,56,59,31$ (we divide 720 by 60 so we can move the ; one place). We repeat this and obtain:

$$12 \times 31;24,56,59,31 = \frac{1}{5} \times 31,24;56,59,31 = 6,16;\ 59,23,54,12 = 360 + 16;59,23,54,12.$$

Here we see what has happened. While 360+16 is of course 376 and this would give us the (correct) number 376; 59, 23, 54, 12, al-Ṭūsī accidentally copied the 16 from his scrap paper in the number in his manuscript. The *khāmisa* can only be explained as a quick and confused count of the number of fractions on the side of al-Ṭūsī. As al-Ṭūsī used the correct number just shortly after, it could be argued that al-Ṭūsī wasn't even consciously aware of his error. In my opinion however, it does not make it legit to emendate the text in this way as it is as it is in all manuscripts that have been studied so far.

[98] Luckey, P., p. 278

One point on the difficulty of these sexagesimal numbers needs to be made before we can move on. This is the number 0; 31, 24, 56, 59, 31 for **h z**, a side of a 720-sided circumscribing polygon. Knorr thinks it actually reads 0; 31, 24, 57, 59, 31 and elaborates in a footnote that this value, if multiplied by 720 gives a different value (namely, 376; 59, 35, 54, ...) then al-Ṭūsī gives. He then even goes so far as to say that if one would calculate backwards, 376; 59, 23, 54, 12 (the number al-Ṭūsī gives for the perimeter of the circumscribing polygon) would imply a starting value of 0; 31, 24, *56*, 59, 31. Now as I've noted before, the difference between six and seven is very ambiguous and if evidence shows that one of the two works better than the other, you can assume that that is the value implied by the author. Here Knorr recalculates the value to 56 instead of 57 but insists that this is due to a scribal error by the computist and even goes so far as reprimanding Luckey and Woepcke for simply transmitting the values and not checking them.[99] It is actually really simple to check whether it should be 56 or 57 and that is to reexamine the computation where the number of **h z** is based on. Al-Ṭūsī computes the number of **h z** by doubling the number of **h e** (page 29). The number of **h e** is 0; 15, 42, 28, 29, 45. We can double each individual number and then readjusting the values to the sexagesimal system. Doing this, we get 0; 30, 84, 56, 58, 90. Readjusting gives 0; 31, 24, *56*, 59, 30. This gives us a second clue that the value should read 56. First it was noted that when Knorr calculated **h z** from the value of the perimeter (calculating backwards) it gave the number 56. Then we saw that if we calculate **h z** from **h e** (calculating forward) it also gave the number 56. I think it is safe to say that, bearing the ambiguity of the reading of the numbers six and seven in mind, the text really reads 56. On a side note, it is peculiar and should be noted as a scribal error that the manuscripts read 31 as the last number while our computation gave 30. Because the two numbers only differ by approximately 1×10^{-9} this is in no way an issue.

At last, al-Ṭūsī makes from these numbers an approximation that looks very similar to the approximation of Archimedes. This is in an unusual form of $3+\frac{10}{70;38,41,21} \approx 3{,}141553196379305$, and $3+\frac{10}{70;37,47,37} \approx 3{,}141583110236151$. Al-Ṭūsī gives as a middle value $3+\frac{10}{70;38,14,29} \approx 3{,}141568151727502$. All these numbers are of the form $3+\frac{10}{70;x}$, with x a fraction. The fraction of the last number is actually the middle value of the other two fractions (so this makes the final approximate not the exact middle value of al-Ṭūsī's upper and lower bound). This last value that al-Ṭūsī states is just approximately $2{,}45 \times 10^{-5}$ (0,0000245) more than the actual value of π. That means that the first four decimals al-Ṭūsī gives are correct. However, with the combination of a couple of scribal (and possibly calculation) errors and the fact that his results rely on a wrong value of the chord of ½°, we can assume that al-Ṭūsī was capable of even more accurate approximations of π.

[99] Knorr, W.R., pp. 593-594, note 57

The argument of the third proposition

Here al-Ṭūsī returns to the original text of Archimedes, following it closely but with one important change. Archimedes opens his proposition by stating that π is equal to $3\frac{1}{7}$ and works the proposition out from this value. Al-Ṭūsī however, merely states that *if* π is equal to $3\frac{1}{7}$ then the following proposition holds true. This is important as π is actually not really equal to $3\frac{1}{7}$ (as was just concluded in al-Ṭūsī's second proposition), rather it is (as al-Ṭūsī states) a ratio commonly used by surveyors. He also elaborates it a little bit more to make it more clear. For example, he doesn't state the result (that the ratio circle:square is 11:14), but also states the ratio 22:28, which is the ratio actually obtained from the proof.

He finishes the text in a formal but simple way. The end doesn't make explicit that it is also the end of the book and this is another indication that al-Ṭūsī did not himself compile all his 'middle books' but that someone else put them all together.

The use of numbers

Throughout the treatise a somewhat consistent writing system for numbers is used. All numbers less than 100 are written in words, this even extents to fractions. Fraction take the morpheme *fuʿl* and complexer fractions take the form 'x parts of y parts'. Other numbers that are written out are 100 and 1000 because they have a proper name. There are two exceptions to this rule. First, in the second proposition, when al-Ṭūsī divides lines to create a figure inscribed in the circle, a certain ratio is simplified by multiplying it by $\frac{11}{40}$. Then 240 becomes 66, and although 66 is less than 100, al-Ṭūsī still writes it in number symbols. It looks like it is done this way to remain coherent with the other ratios and it is a lot more readable and understandable this way. The other exception happens just after that. It is near the end of the second proposition but before the astronomical intermezzo, just before the final conclusion. Al-Ṭūsī wants to say that 6336 is more than $2017\frac{1}{4} \times 3\frac{10}{71}$. While he just wrote 2017 in number symbols and ¼ with a word a couple of lines before, this time he writes 2017¼ completely in words (using the dualis of 1000, *alfayn*).

The sexagesimal system with its letter combinations for representing numbers is something completely different and has already been discussed above. A note may be added for the observant reader who may have noticed that here and there a *hā* (ه) is used to indicate zero. It is consistently used at the beginning of a number if that number is below one, with the exception of **d̲ g** on page 29. The numbers at the end are also without a *hā* but they belong actually to 376 and $3\frac{10}{71}$ respectively so in theory they do not need a *hā*. With the numbers that belong to 376, it is also used at the end of a number. This can be seen as a way to make the numbers more coherent and comparable (because now they both consist of four fractions).

The use of language

The use of language is very consistent and pretty much according to the rules of Classical Arabic. The most eye-catching of this is the consisting use of the construct state of a word before letters used in the geometrical figures, to indicate it is in a genitive construction. In most cases this just means the article is missing but for example in al-Ṭūsī's second proposition it is stated that "If we add up the numbers [that belong to] **h e** and **e g**..." Here 'numbers' is in dual to indicate it refers to two numbers and because it is in construct state it is not *ᶜadadayn* but *ᶜadaday*, dropping the *–n* at the end. This is also a good example of the consistent usage of singular, dual and plural form of words, whenever there is a reference to two objects, the dual is used. Another good example of this usage of the dual that makes it actually very good to understand the sentence is again in al-Ṭūsī's second proposition. When the inner polygon is being defined, right at the beginning when all kinds of equalities are established, al-Ṭūsī states "and the two angles [in] **h** [and] **b** are right." Here *zāwiyatā* is used, clearly distinct from the singular *zāwiya* and the plural *zawāyā*. In this way it is immediately clear that two the angles *at* **h** and **b** are meant, not for example some angle that involved the line from **h** to **b**.

There are not a lot of peculiarities in the text but I will make mention of two. The first one is on page 23. The translation reads "The diameter is in this measure 4673½ ...". While the Arabic literally reads "The diameter is in this measure twice 4673½ ...". This 'twice' (*ḍaᶜf*) seems completely out of place and that is also why I did not included it in the translation (although both the Hyderabad edition and the Tehran facsimile edition have it). The other peculiarity is on page 26. Al-Ṭūsī consistently uses *fī* ('in') to define a polygon inscribing the circle and *ᶜalā* ('on') to define a polygon circumscribing the circle. However, where the translation reads "the polygon of 96 sides inscribed in the circle is 6336 ...", *ᶜalā* is used where *fī* is expected. Again there is no reasonable explanation and so we can only but neglect it and read it as if it indicates that the polygon is inscribing the circle.

A smaller note is that in a couple of cases al-Ṭūsī states *aᶜnī* meaning 'I mean'. This resonates the nowadays widely used stopping phrase *yaᶜnī*, which translates into English as 'like, you know...'. I think it is surprising to see such ordinary Arabic here in written form. Overall al-Ṭūsī leaves a convincing impression behind on the reader of his knowledge of Arabic grammar and vocabulary.

Chapter 4

Concluding remarks

A whole array of topics have been discussed in this thesis so a short summary of some important results this thesis puts forward seems in place. First, there was the question of what exactly this term *mutawassiṭāt* is. By now, it is clear that the *mutawassiṭāt* is a collective name for Arabic translations of Greek mathematical treatises and Arabic mathematical treatises that are to be read between Euclid's *Elements* and Ptolemy's *Almagest*. There is no definitive list of books that are in this collection and books that are not, rather, it is a more fluid notion that came to use from at least the 10th century C.E. Before al-Ṭūsī, at least al-Nasawī can be named as someone who edited one or more 'middle books', but it seems that only after al-Ṭūsī came the 'middle books' to be some sort of 'middle book', with each treatise as a separate chapter (it is important to note that there is even among the manuscripts of al-Ṭūsī's *taḥrīr al-mutawassiṭāt* differences in contents, some of the fluidity remained).

Another important question is why then did al-Ṭūsī edit these texts and include valuable remarks in it. In al-Ṭūsī's own words, this was to improve the corrupted versions that were in use and make them more understandable. This is true, but taking into account the editorial decisions that improved the text significantly and the addition of valuable new material to the treatises we should feel more inclined to conclude that al-Ṭūsī really did his best to replace all other editions that were around and to establish one good version, bundled together with other treatises that belong to the 'middle books'. From the completing dates of the different treatises we can also see that this was a long term project, and as there is no dedication to somebody (the one who ordered the edition) we can deduce that al-Ṭūsī did this in his own time. This shows that he enjoyed mathematics for its own sake and felt the urge to enable other people to share the same passion.

In the edition of *The measurement of the circle* a couple of significant and original contributions of al-Ṭūsī should be noted. First there is of course the rearrangement of the propositions. This is significant in that al-Ṭūsī chose not to make an edition that would be close to Archimedes' text, but one that was logically correct. Secondly, at the end of the first proposition there is a generalization that is completely original. Instead of staying with the surface area of a circle, al-Ṭūsī shows that this is only a special case and that the reasoning just as well proves the more general case of the surface area of a sector (a 'piece of pie'). Thirdly, and most importantly, there is of course the highly original addition to al-Ṭūsī's second proposition. Here, al-Ṭūsī uses a number from a trigonometric table to compute the circumference of a regular polygon that is inscribed in a circle and one that is circumscribed, making two 720-sided polygons. From these two values al-Ṭūsī deduces an approximation of π that is in modern terms just 0,0000245 off, although it must be said that the manuscripts contain scribal errors so it could have been originally an even closer approximation. Fourthly, al-Ṭūsī seemed to sense the error of the statement that π is equal to $3\frac{1}{7}$ (this is asserted in Archimedes' third proposition and is needed to complete the proof of the proposition) and corrected this into an if clause, remarking that this is an approximation regularly used by surveyors. These are all valuable additions to Archimedes' text and shows that al-Ṭūsī did not just wanted to render the text as it is, but truly wanted to make it more understandable and more comprehensive.

Al-Ṭūsī's edition of *The measurement of the circle* proved to be a very good study subject to dive into the vast area of the History of Arabic Sciences. It enabled me to study a whole range of issues at the same time, while remaining small enough to be handled in a bachelor thesis. Especially the concept of *mutawassiṭāt* was intriguing and deserves more attention in the future I think. I was also really impressed at the richness of the material, and with that I especially mean the big efforts al-Ṭūsī and the scribes after him made to complete and distribute this, in its own respect purely mathematical, treatise. Much needs to be done to uncover this richness and therefore I want to conclude with some final words on further research.

Notes on further research

This thesis also showed the severe understudied situation of this subject. Only a handful of scholars have devoted time on this subject and of them Steinschneider proved to be one of the best sources, while his study dates all the way from 1865. The more contemporary scholarship from Knorr (1989) was shown to be incorrect on several occasions. The latest research on the topic is an article published in 2000,[100] in which Kheirandish reports on the various manuscripts that lie around in Iranian libraries, ready to be read. Only her 3-page long account already reveals some interesting points, for example, the inclusion of al-Ṭūsī's famous essay on Euclid's parallel postulate and *Correction on Optics* from al-Kindī in the 'middle books'. Even more interesting, the date of completion of the whole manuscript is 671/1273 is just before al-Ṭūsī's death. If these manuscripts would be studied more in-depth, surely a better understanding of the contents of the 'middle books' and also the purpose of al-Ṭūsī would become much more clear. A rigorous assessment of the number, contents and origination of 11th-17th century manuscripts is also much needed as it could give clues to what extend al-Ṭūsī's text was used and in what kind of context. All in all, much remains to be done and for a fair picture of the history of science there is also a very clear relevancy to do this.

[100] Kheirandish, E., *A Report on Iran's 'Jewel' Codices of Ṭūsī's* Kutub al-Mutawassiṭāt, pp. 131-144 of: Pourjavady, N., Vesel, Ž., *Naṣīr al-Dīn Ṭūsī Philosophe et Savant du XIIIe Siècle*, Presses Universitaires D'Iran, Téhéran, 2000

References

- aṭ-Ṭūsī, Naṣīraddīn, "Kitāb al-Kura wa-l-usṭuwāna li-Arkhimīdis [Archimedes] bi-taḥrīr", *majmū*[c] *al-rasā'il* (2 vols.), Ḥaydarābād, 1359/1940, pp. 127-133. Reprinted in: Sezgin, F., *A Collection of Mathematical and Astronomical Treatises as Revised by Naṣīraddīn aṭ-Ṭūsī*, Volume 48 of *Islamic Mathematics and Astronomy*, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp. 377-389 (٣٧٧ – ٣٨٩)
- al-Tusi, Nasir al-Din, *Taḥrīr-e Mutawassīṭāt*, Introduction by Dr. Jafar Aghayanī-Chavoshī, Insitute for Humanities and Cultural Studies, Tehran, 2005
- Badakhchani, S.J., *Contemplation and Action*, I.B. Tauris, London, 1998
- Clagett, M., *Archimedes in the Middle Ages*, Vol. I The Arabo-Latin Tradition, The University of Wisconsin Press, Madison, 1964
- Dijksterhuis, E.J., trans. by C. Dikshoorn, *Archimedes*, Princeton University Press, New Jersey, 1987
- EI^2 = *Encyclopedia of Islam,* second edition, Brill, Leiden
- Gutas, D., *Avicenna and the Aristotelian Tradition*, E.J. Brill, Leiden, 1988
- Heath, T., *A History of Greek Mathematics*, vol. II, Clarendon Press, Oxford, 1965
- Kheirandish, E., *A Report on Iran's 'Jewel' Codices of Ṭūsī's* Kutub al-Mutawassiṭāt, pp. 131-144 of: Pourjavady, N., Vesel, Ž., *Naṣīr al-Dīn Ṭūsī Philosophe et Savant du XIII*[e] *Siècle*, Presses Universitaires D'Iran, Téhéran, 2000
- Knorr. W.R., *Textual Studies in Ancient and Medieval Geometry*, Birkhäuser, Boston, 1989, pp. 535-594
- Krause, M., "Stambuler Handschriften islamischer Mathematiker", *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* (Berlin), Abteilung B: Studien 3. 1936, pp. 437-532. Reprinted in: Sezgin, F., *Miscellaneous Texts and Studies on Islamic Mathematics and Astronomy VIII*, vol. 83 of Islamic Mathematics and Astronomy, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp.237-332
- Luckey, P., *Der Lehrbrief über den Kreisumfang (ar-Risāla al-Muḥīṭīya) von Gamšid b. Mas*[c]*ūd al-Kāšī übersetzt und erläutert*. Herausgegeben von Alfred Siggel,Akademie Verlag, Berlin, 1953. Reprinted in: Sezgin, F., *Al-Kāsh Texts and Studies*, Islamic Mathematics and Astronomy, vol. 56, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1998, pp. 227-329
- Muwafi, A., Philoppou, A.N., "Appendix to 'An Arabic Version of Eratosthenes on Mean Proportionals'", *Journal of the History of Arabic Science* 5. Aleppo, 1998, pp. 145-165. Reprinted in: Lorch, R., *Arabic Mathematical Sciences*, Variorum, Aldershot, 1995, part IV: *A Note on the Technical Vocabulary in Erastosthenes' Tract on Mean Proportionals*
- Ragep, F.J., *Naṣīr al-Dīn al-Ṭūsī's* memoir on astronomy, Springer-Verlag, New York, 1993

- Rashed, R., *Al-Kindī's Commentary on Archimedes' 'The Measurement of the Circle'*, pp. 7-54 in: *Arabic sciences and Philosophy*, vol. 3, Cambridge University Press, Cambridge, 1993
- Rosenfeld, B.A., Ihsanoglu, E., *Mathematicians, Astronomers, and other scholars of Islamic civilization and their works (7th-19th c.)*, Research Centre for Islamic History Art and Culture, Istanbul, 2003
- Sezgin, F., *Geschichte des Arabischen Schrifttums*, Band V Mathematik bis ca. 430H, Brill, Leiden, 1974
- Steinschneider, M., "Die mittleren Bücher der Araber und ihre Bearbeiter", *Zeitschrift für Mathematik und Physik*, nr. 10, Leipzip, 1865, pp. 456-498. Reprinted in: Sezgin, F., *Euclid in the Arabic Tradition Text and Studies I*, Islamic Mathematics and Astronomy, vol. 17, Institute for the History of Arabic-Islamic Science, Frankfurt am Main, 1997
- Thomas, I., *Selections Illustrating the History of Greek Mathematics*, vol. I From Thales to Euclid, Cambridge University Press, Cambridge, Massachusetts, 1967
- Toomer, G.J., *Conics. Books V to VII : the Arabic translation of the lost Greek original in the version of the Banū Mūsā*, Springer Verlag, New York, 1990

Appendix: table of contents of the Tehran facsimile edition

By Prof. Dr. J.P. Hogendijk[101]

List of the *Middle Books* in the Recension of Naṣīr al-Dīn al-Ṭūsī in the Facsimile Edition of Ms. Tabriz, Melli Library, no. 3484, edited by Dr. Jafar Aghayani Javoshi (Tehran, institute for Humanities and Cultural Studies, 2005), with references to vols. 5 and 6 of GAS = F. Sezgin, *Geschichte des arabischen Schrifttums* (Leiden 1974-1978).

1. p. 1-22, Euclid, *Data*, GAS V, 116.

2. p. 27-28, 25-26, 23-24, gap, 29-56, Theodosius, *Spherics*, GAS V, 154.
The gap begins with *Spherics*
book I, the end of prop. 19 and ends with *Spherics*, Book II, beginning of prop. 8.

3. p. 58-63, Autolycus, *Moving Sphere*, GAS V, 82.

4. p. 66-84. Euclid, *Optics*, GAS V, 117

5. p. 88-95, Theodosius, *Inhabited Places*, GAS V, 155.

6. p. 98-117, Autolycus, *Risings and Settings*, GAS VI, 73.

7. p. 122-145, Euclid, *Phaenomena*, GAS V, 118.

8. p. 147-168, Theodosius, *Days and Nights*, GAS V, 156, dated.

9. p. 171-184, Aristarchus, *Sizes and Distances of the Sun and Moon*, GAS VI, 75, dated.

10. p. 187-189, Hypsicles, *Ascensions*, GAS V, 145.

11. p. 192-203, "Archimedes", *Lemmata*, GAS V, 131.

12. p. 205-214, Thābit ibn Qurra, *Assumed Things* (Mafrūḍāt), GAS V, 271 no. 19.

13. p. 221-331, Menelaus, *Spherics*, GAS V, 162 no. 5. The last three pages (328-331) are not found in the Hyderabad edition.

14. p. 331-332, Ibn al-Haytham, *Division of the Line which Archimedes used in the second Book On the Sphere and Cylinder*. GAS V, 371 no. 31.

15. p. 335-442, Naṣīr al-Dīn al-Ṭūsī, *On the Transversal Theorem*. Rosenfeld and Ihsanoglu p. 214 no. M 14.

16. p. 447-532, Archimedes, *On the Sphere and Cylinder*, with the commentary of Eutocius, GAS V, 129b.

17. p. 532-541, Abū Sahl Kūhī, *Additions to the Book On the Sphere and Cylinder of Archimedes*, GAS V, 320 no. 25.

18. p. 541-545, Archimedes, *Measurement of the Circle*, GAS V, 130 no. 2.

[101] I thank Prof. Hogendijk wholeheartedly for allowing me to include this table of contents (which is not included in the facsimile edition).